

15-826: Multimedia (Databases) and Data Mining

Lecture #21: DSP tools –
DFT – Discrete Fourier Transform
C. Faloutsos



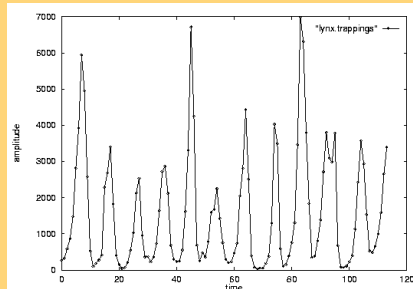
Problem

Goal: given a signal (eg., sales over time and/or space)

Q: Find patterns and/or compress



count



year



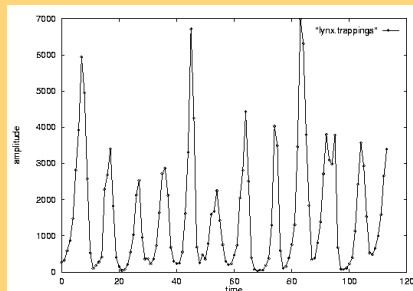
Solutions:

Goal: given a signal (eg., sales over time and/or space)

Q: Find patterns and/or compress

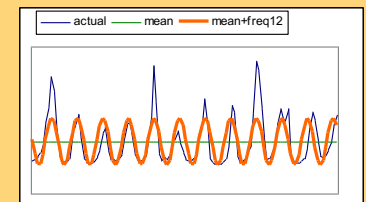


count

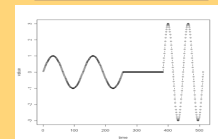
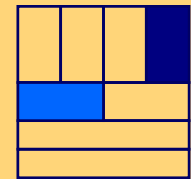


year

A1: Fourier (DFT)



A2: Wavelets (DWT)





Summary of properties

- ✓ Time shift $\rightarrow$ same amplitude spectrum
- ✓ Single/dominant sinusoid: $\rightarrow$ spike in spectrum
- ✓ DFT preserves energy (Parseval's thm)
- ✓ Spike in time: $\rightarrow$ all frequencies
- ✓ Step/Trend/sawtooth: $\rightarrow$ ringing ($\sim$ all frequencies)

Must-read Material

- DFT/DCT: In [PTVF](#) ch. 12.1, 12.3, 12.4; in [Textbook](#) Appendix B.
- Wavelets: In [PTVF](#) ch. 13.10; in [MM](#) [Textbook](#) Appendix C



Outline

Goal: ‘Find **similar / interesting** things’

- Intro to DB

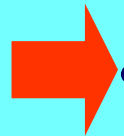
- Indexing - similarity search

- Data Mining



Indexing - Detailed outline

- primary key indexing
- ..



- Multimedia –
 - Digital Signal Processing (DSP) tools
 - Discrete Fourier Transform (DFT)
 - Discrete Wavelet Transform (DWT)



DSP - Detailed outline

- DFT



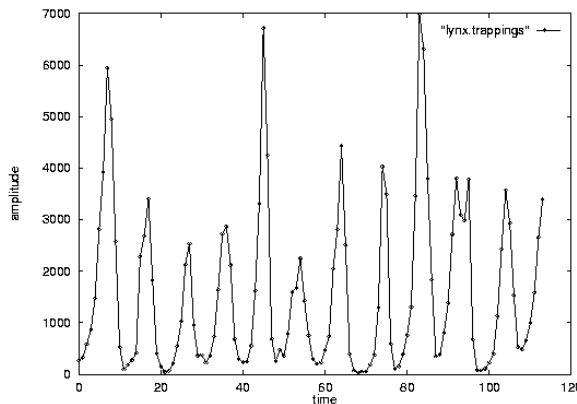
- what
- why
- how
- Arithmetic examples
- properties / observations
- DCT
- 2-d DFT
- Fast Fourier Transform (FFT)

Introduction

Goal: given a signal (eg., sales over time and/or space)

Find: patterns and/or compress

count



lynx caught per year



year

What does DFT do?

A: highlights the periodicities

Why should we care?

A: several real sequences are periodic

Q: Such as?

Why should we care?

A: several real sequences are periodic

Q: Such as?

A:

- sales patterns follow seasons;
- economy follows 50-year cycle
- temperature follows daily and yearly cycles

Many real signals follow (multiple) cycles

Why should we care?

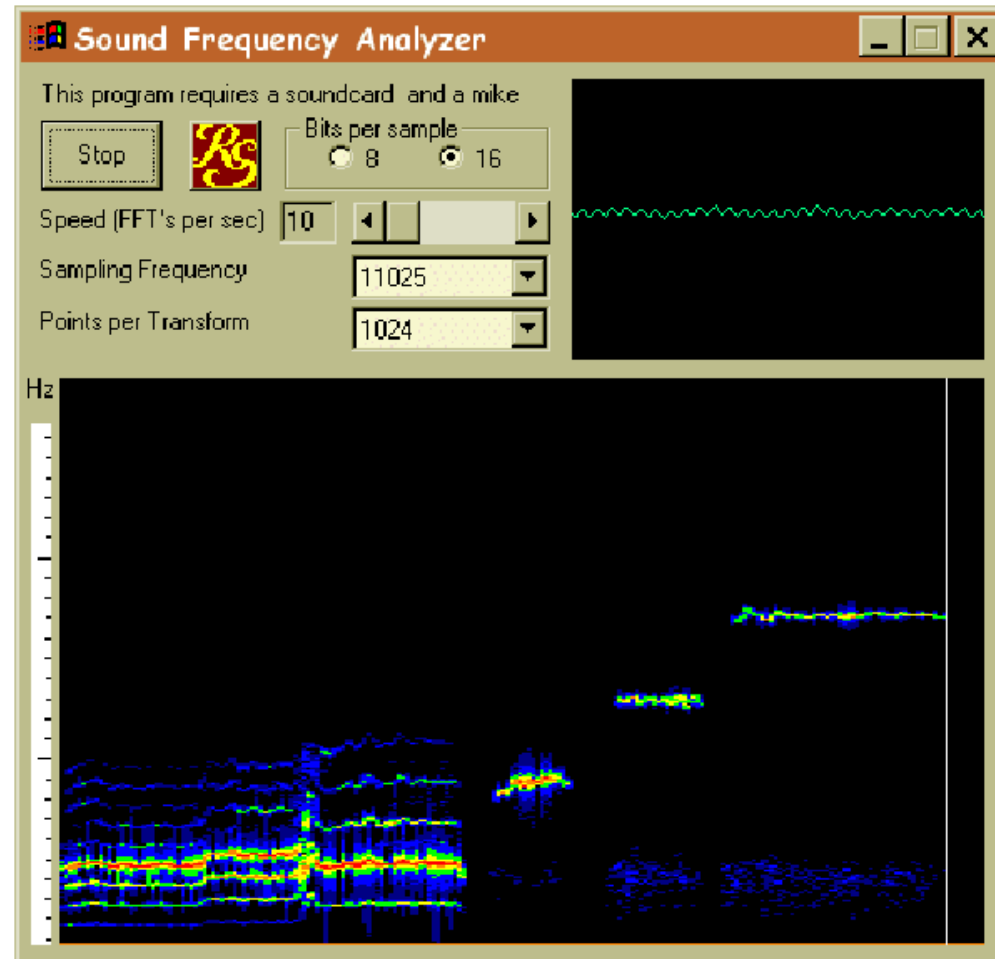
For example: human voice!

- Frequency analyzer

<http://www.relisoft.com/freeware/freq.html>

- speaker identification
- impulses/noise -> flat spectrum
- high pitch -> high frequency

'Frequency Analyzer'

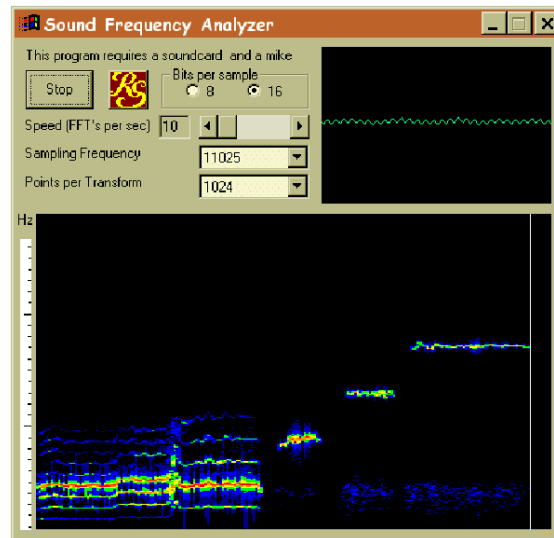


↑
frequency

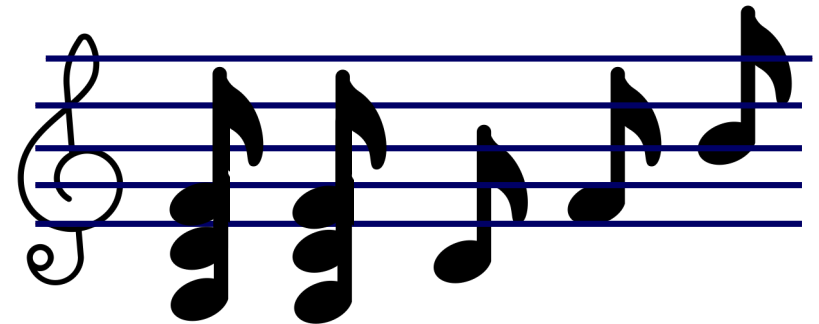
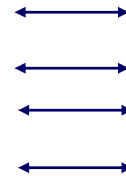
time →

'Frequency Analyzer'

↑
frequency



time →



DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi f t / n) \quad f = 0, \dots, n-1$$

$$(j = \sqrt{-1})$$

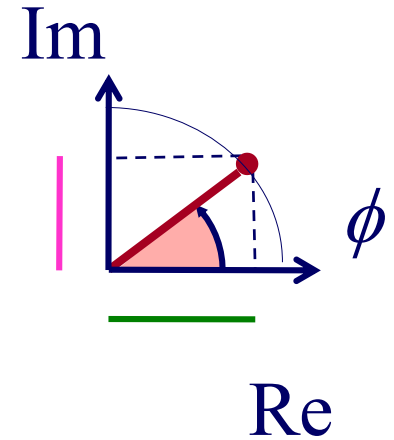
$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j2\pi f t / n)$$

inverse DFT



(Reminder)

$$\exp(\phi * j) = \underbrace{\cos(\phi)} + j * \underbrace{\sin(\phi)}$$



(fun fact: the equation with the 5 most important numbers:

$$e^{j\pi} + 1 = 0$$

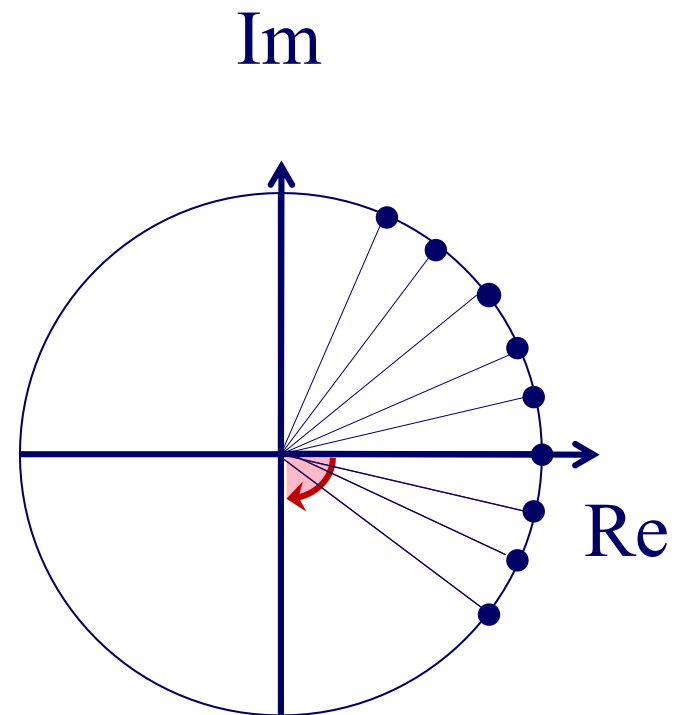
)

Intuition: ‘phasors’

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2 \pi f t / n)$$



Rotation
as $t=0, \dots$



Intuition: ‘phasors’

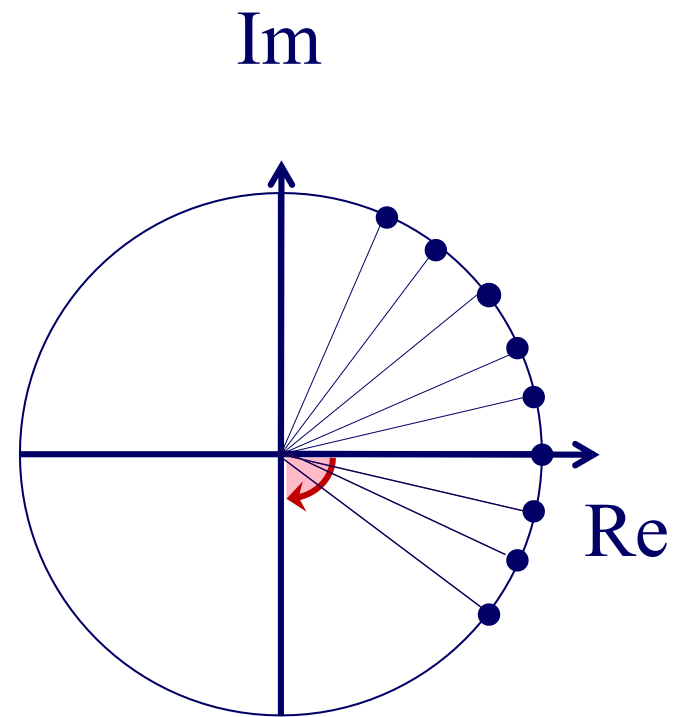
$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2 \pi f t / n)$$



Rotation
as $t=0, \dots$

Real $\rightarrow \cos()$

Im $\rightarrow \sin()$



Intuition: ‘phasors’

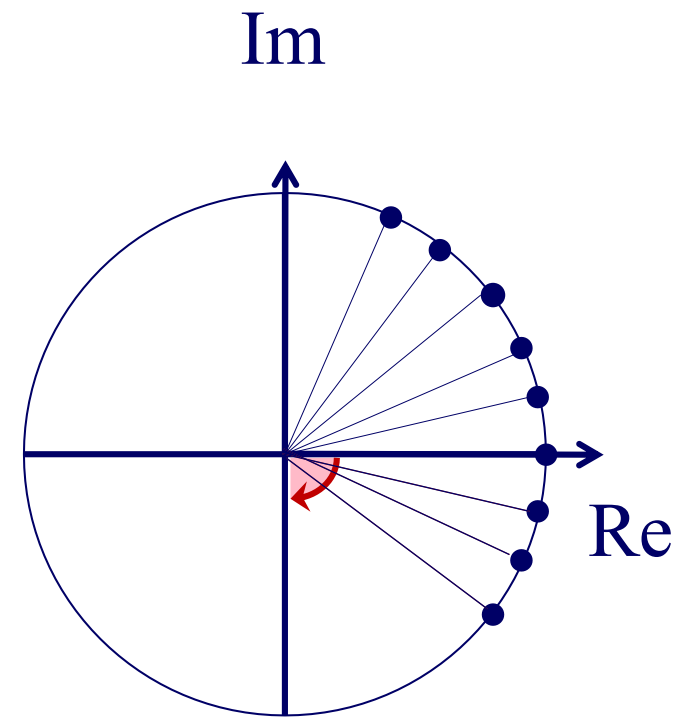
$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2 \pi f t / n)$$



Rotation
as $t=0, \dots$

X_f : how similar is your signal to

- $\cos(-2 \pi f t / n) \rightarrow \text{Re}(X_f)$
- $\sin(-2 \pi f t / n) \rightarrow \text{Im}(X_f)$



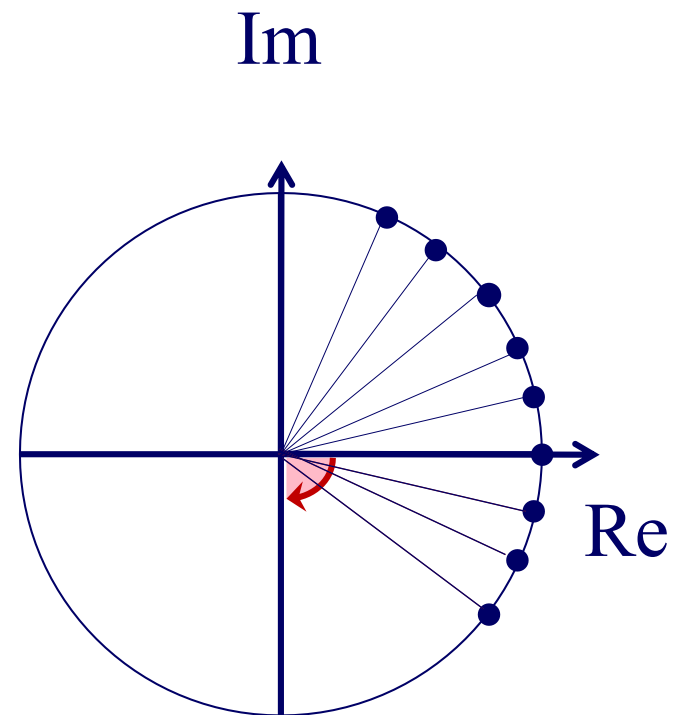
Drill:

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2 \pi f t / n)$$



Rotation
as $t=0, \dots$

$$X_0 = ??$$



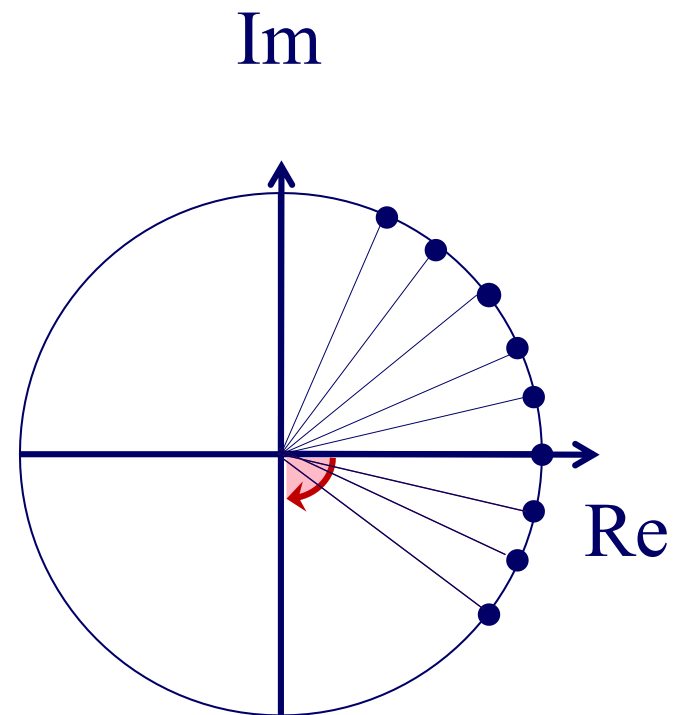
Answer:

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2 \pi f t / n)$$



Rotation
as $t=0, \dots$

$X_0 \propto \text{sum}$



DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in ‘mathematica’



```
x = [1,2,1,2];
```

```
X = Fourier[x];
```

```
Plot[ Abs[X] ];
```

DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in python

```
x = [1,2,1,2]
```

```
f = scipy.fft.fft(x)
```

```
matplotlib.pyplot.plot( ..., abs(f))
```

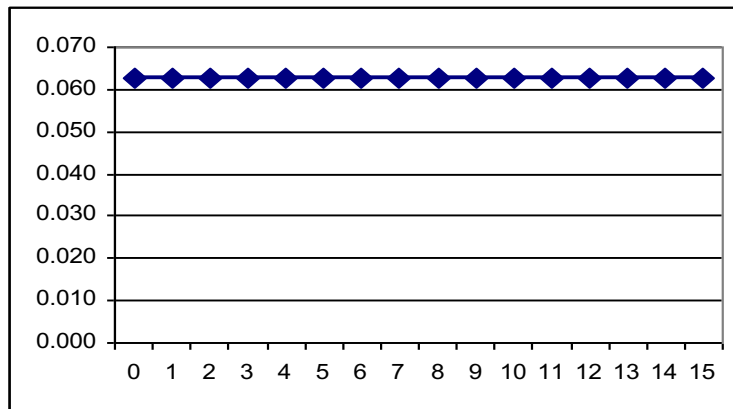


DFT: examples

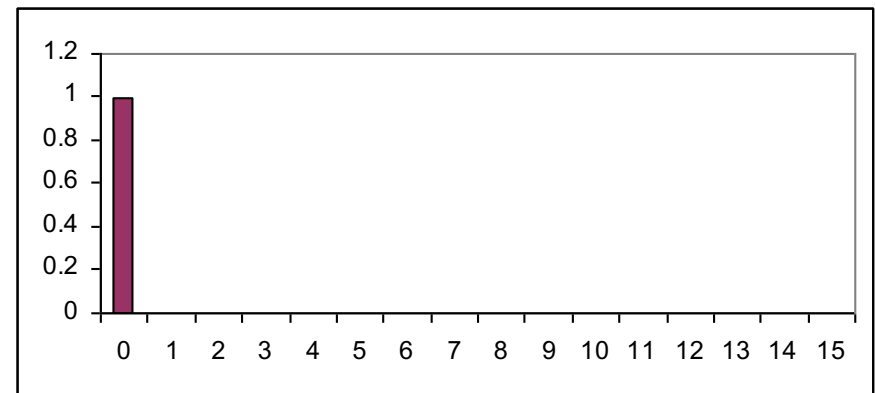
Plot[Abs[Fourier[x]]];

flat

Amplitude



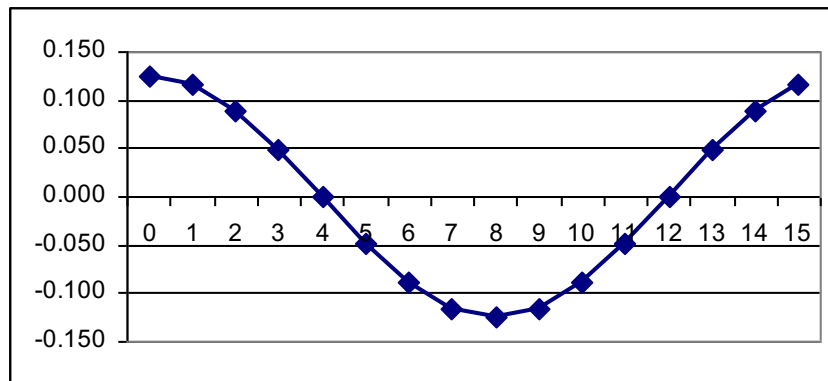
time



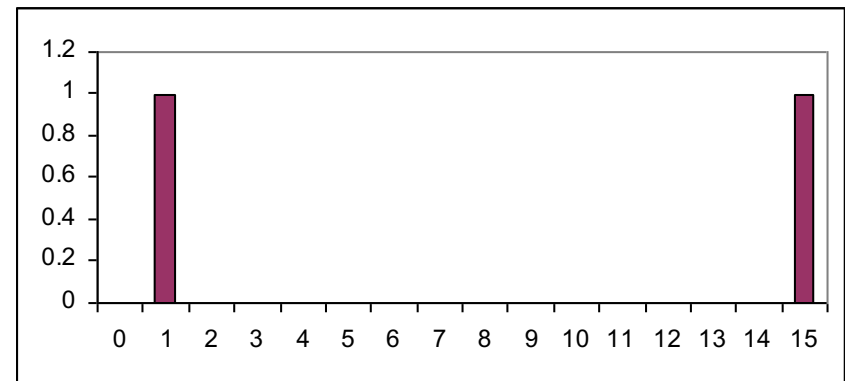
freq

DFT: examples

Low frequency sinusoid



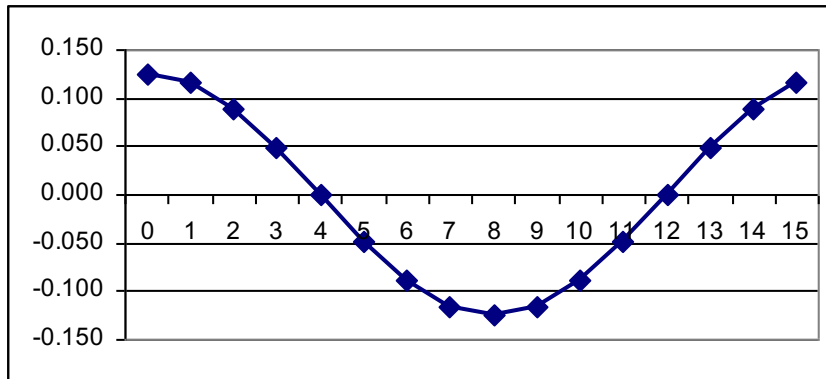
time



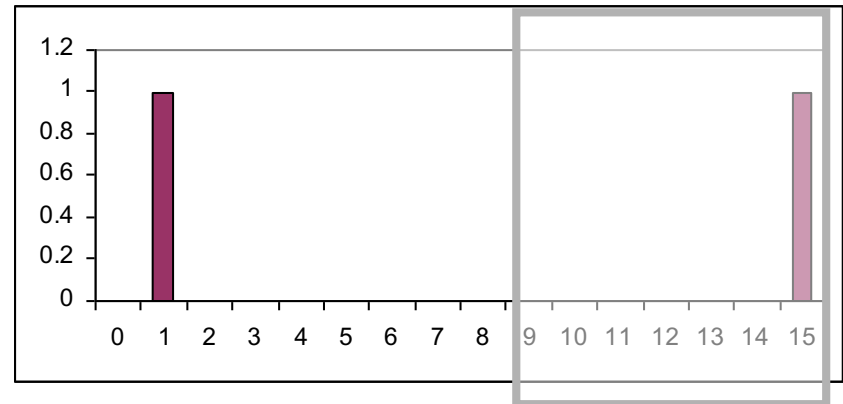
freq

DFT: examples

- Sinusoid - symmetry property: $X_f = X_{n-f}^*$



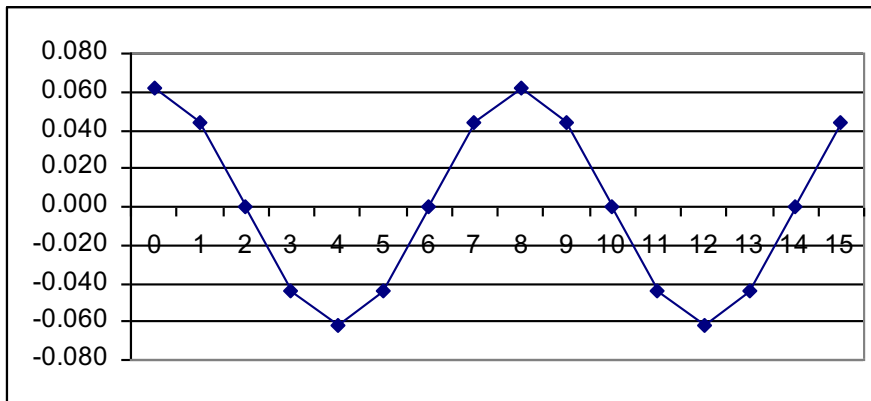
time



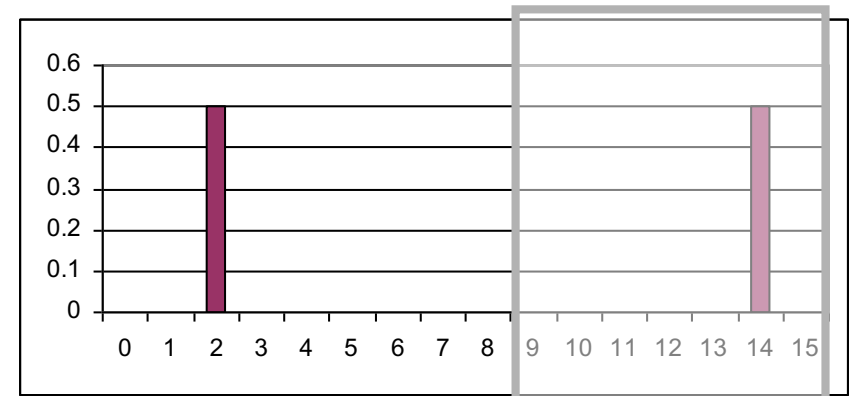
freq

DFT: examples

- Higher freq. sinusoid



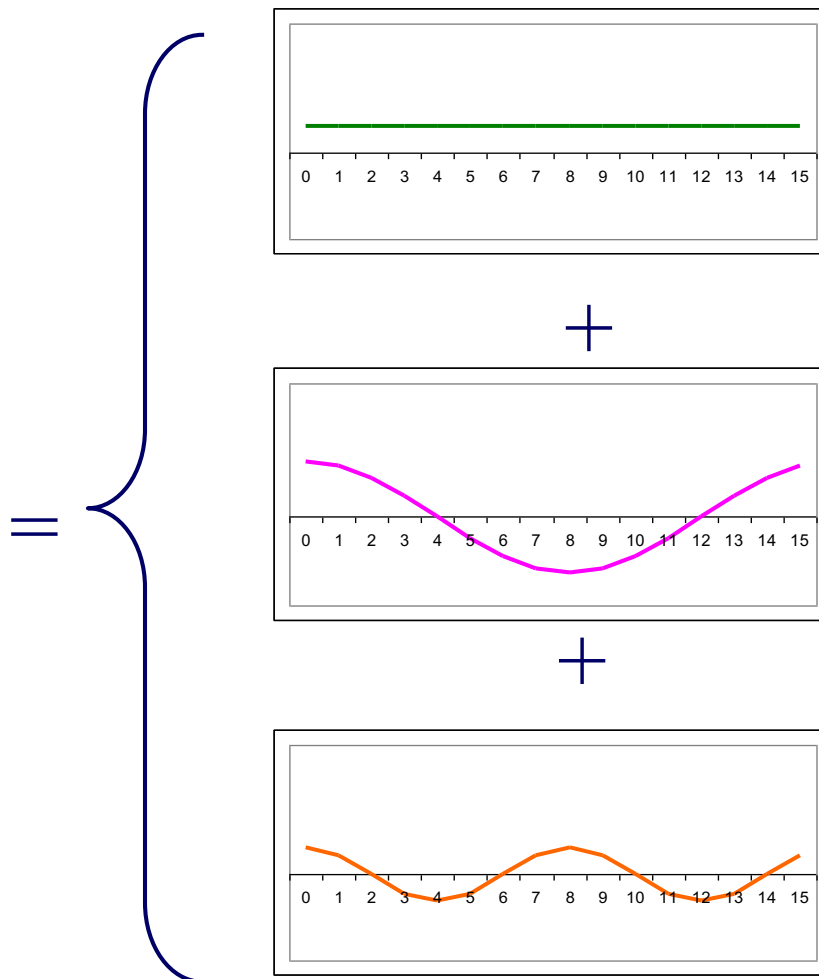
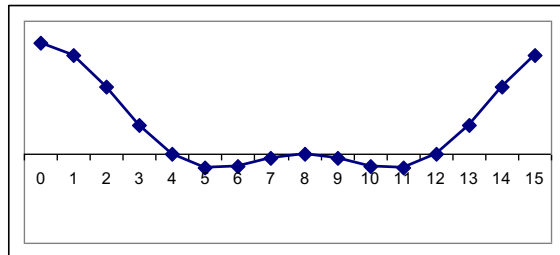
time



freq

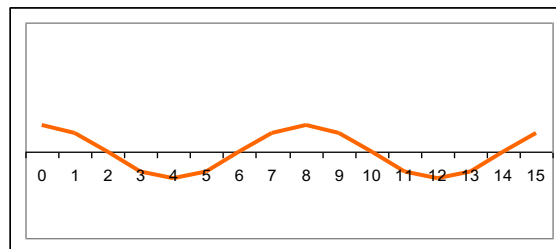
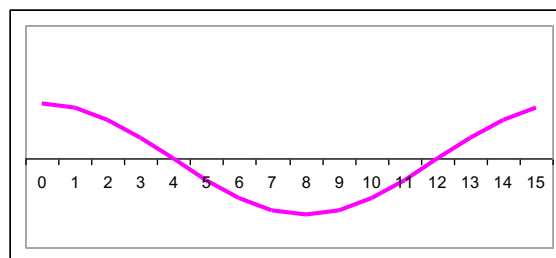
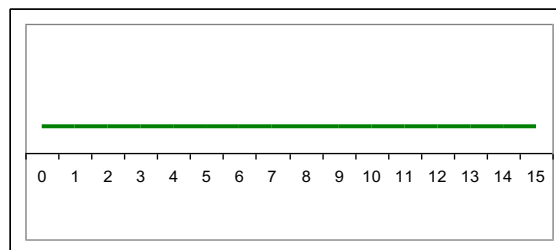
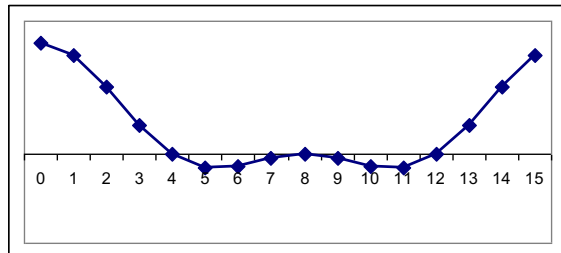
DFT: examples

examples

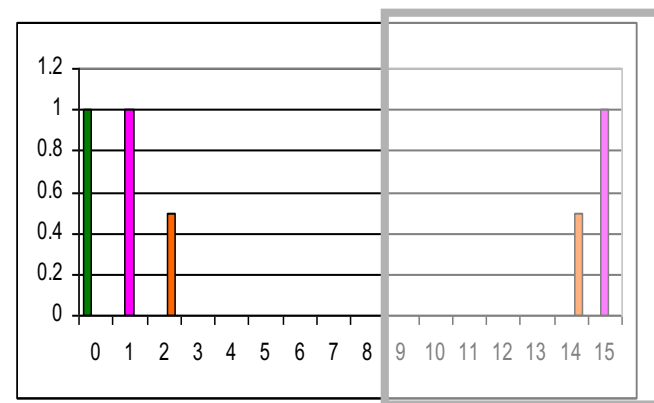


DFT: examples

examples



Ampl.



Freq.

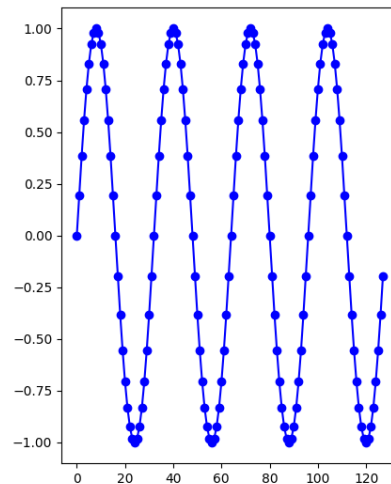
DFT: drills (1 of 2)

- Q1) How does $\sin(2 \pi 4 t / 128)$ ($t = 0 \dots 127$) look like?
- Q2) How about its Fourier spectrum?

DFT: drills (1 of 2)

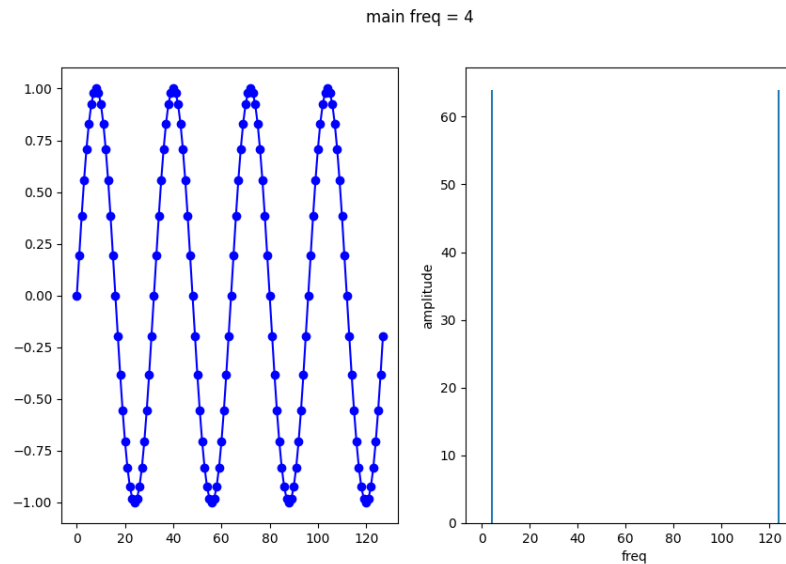
- Q1) How does $\sin(2 \pi 4 t / 128)$ ($t = 0 \dots 127$) look like?
- Q2) How about its Fourier spectrum?

main freq = 4



DFT: drills (1 of 2)

- Q1) How does $\sin(2 \pi 4 t / 128)$ ($t = 0 \dots 127$) look like?
- Q2) How about its Fourier spectrum?



DFT: drills (2 of 2)

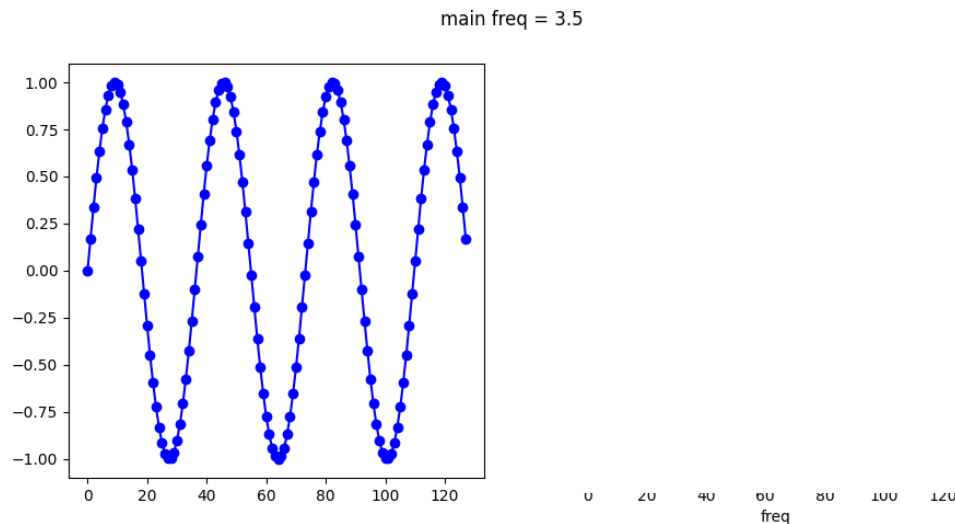


- Q1) How does $\sin(2 \pi 3.5 t / 128)$ ($t = 0 \dots 127$) look like?
- Q2) How about its Fourier spectrum?

DFT: drills (2 of 2)



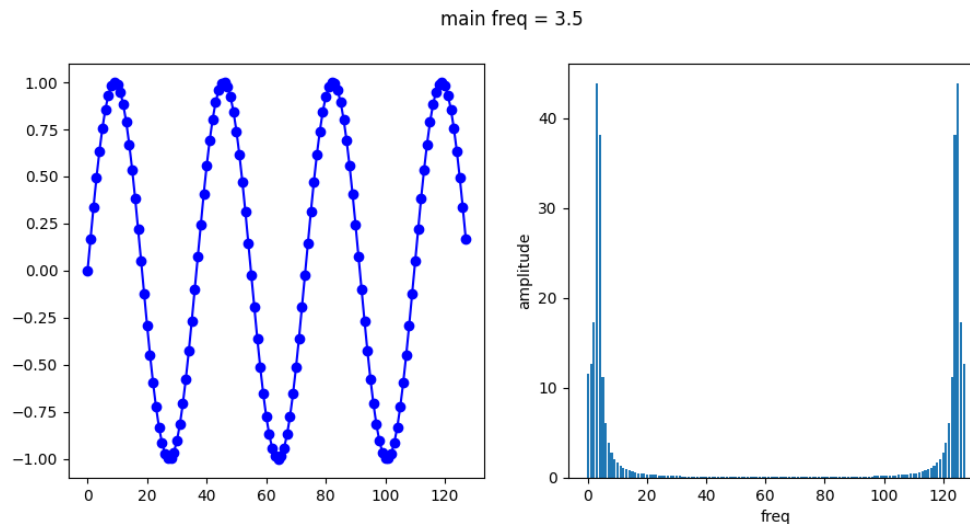
- Q1) How does $\sin(2 \pi 3.5 t / 128)$ ($t = 0 \dots 127$) look like?
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DFT: drills (2 of 2)



- Q1) How does $\sin(2 \pi 3.5 t / 128)$ ($t = 0 \dots 127$) look like?
- Q2) How about its Fourier spectrum?





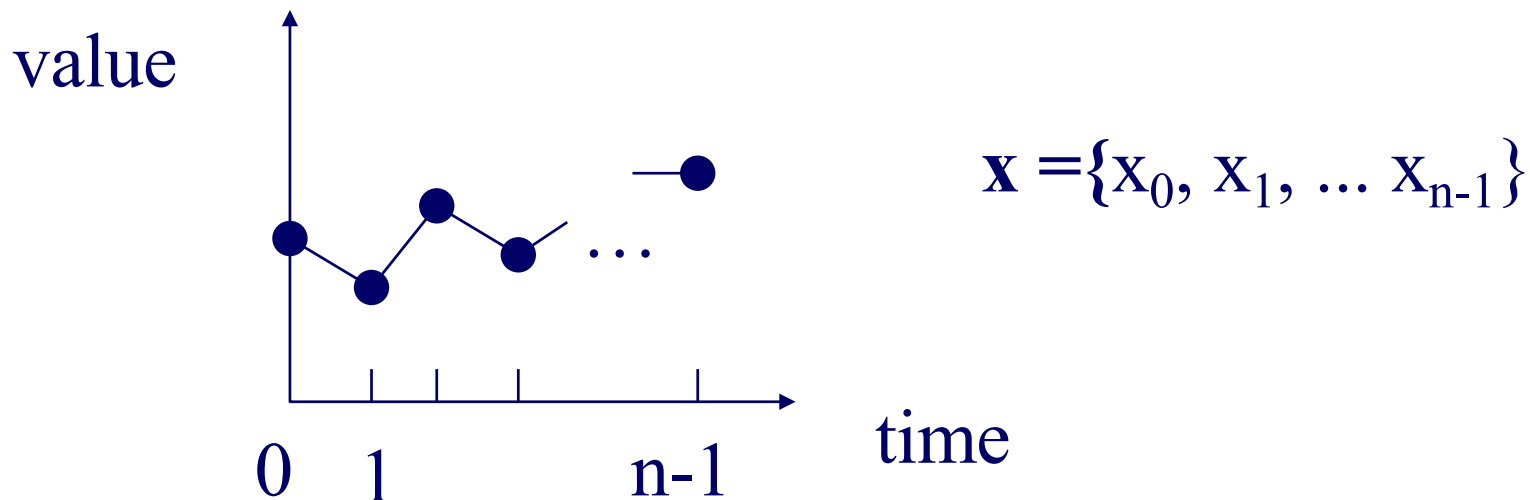
DSP - Detailed outline

- DFT
 - what
 - why
 - – how
 - Arithmetic examples
 - properties / observations
 - DCT
 - 2-d DFT
 - Fast Fourier Transform (FFT)

How does it work?

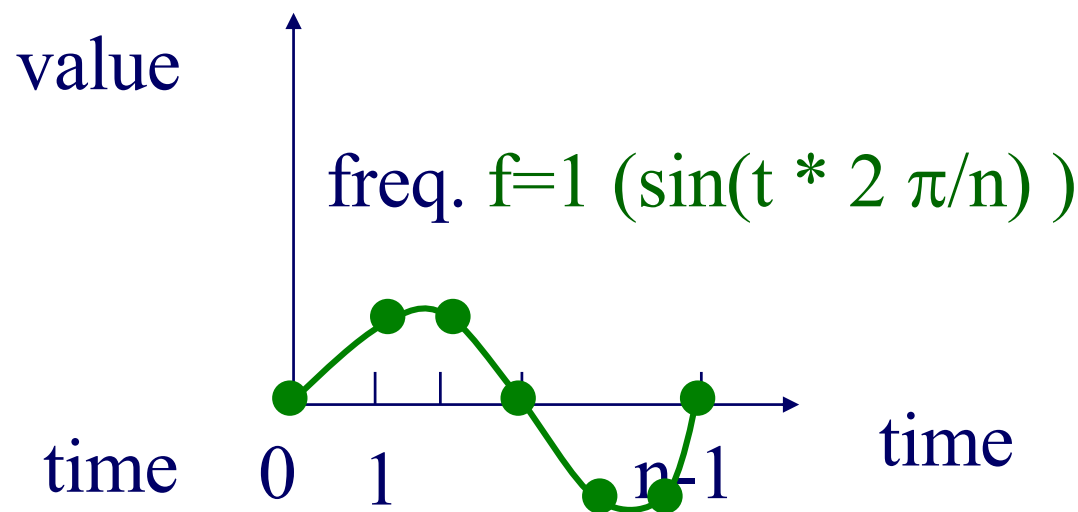
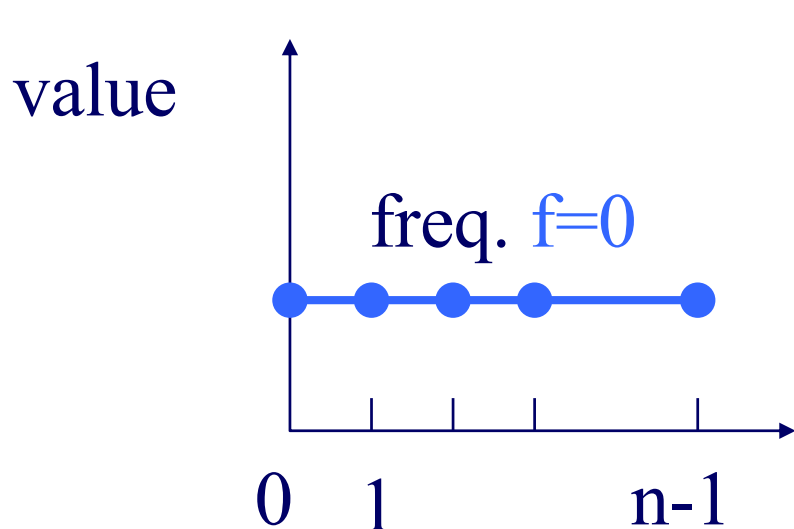
Decomposes signal to a sum of sine (and cosine) waves.

Q: How to assess ‘similarity’ of $\mathbf{x}$ with a wave?



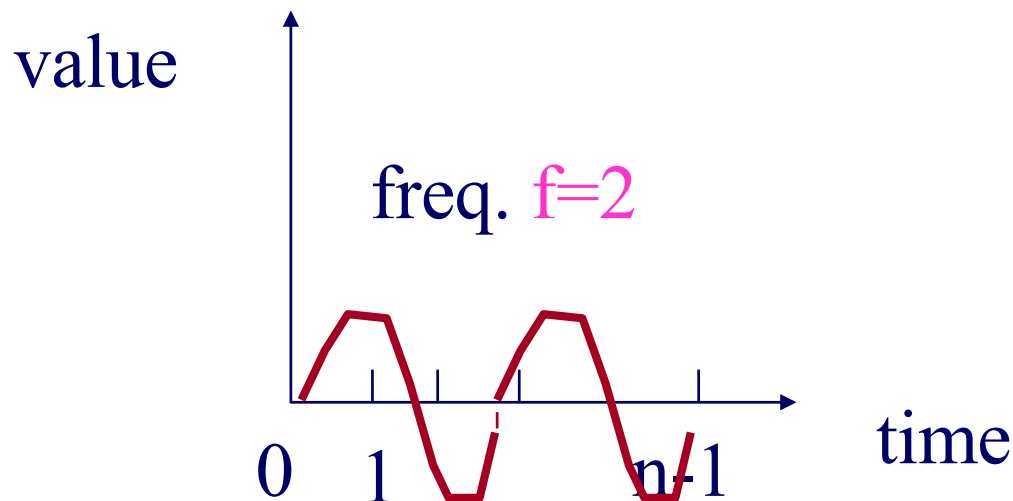
How does it work?

A: consider the waves with frequency 0, 1, ...;
use the inner-product ($\sim$ cosine similarity)



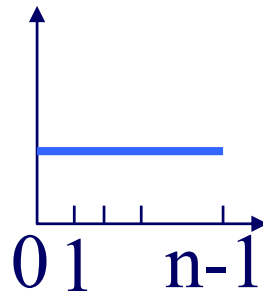
How does it work?

A: consider the waves with frequency $0, 1, \dots$;
use the inner-product ($\sim$ cosine similarity)

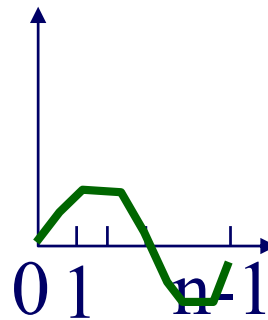


How does it work?

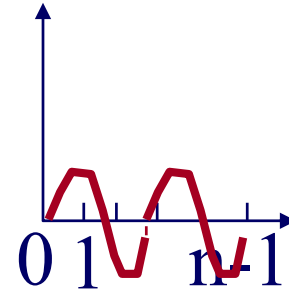
'basis' functions
(vectors)



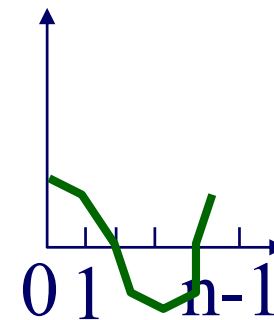
sine, freq = 1



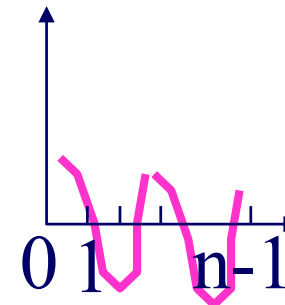
sine, freq = 2



cosine, $f=1$



cosine, $f=2$



How does it work?

- Basis functions are actually n -dim vectors, **orthogonal** to each other
- ‘similarity’ of $\mathbf{x}$ with each of them: inner product
- DFT: $\sim$ all the similarities of $\mathbf{x}$ with the basis functions

DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in ‘mathematica’

```
x = [1,2,1,2];
```

```
X = Fourier[x];
```

```
Plot[ Abs[X] ];
```

DFT: definition

(variations:

- $1/n$ instead of $1/\sqrt{n}$
- $\exp(-\dots)$ instead of $\exp(+\dots)$

DFT: definition

Observations:

- X_f : are complex numbers except
– X_0 , who is real
- $\text{Im}(X_f)$: $\sim$ amplitude of sine wave of frequency f
- $\text{Re}(X_f)$: $\sim$ amplitude of cosine wave of frequency f
- $\mathbf{x}$: is the sum of the above sine/cosine waves

DFT: definition

Observation - SYMMETRY property:

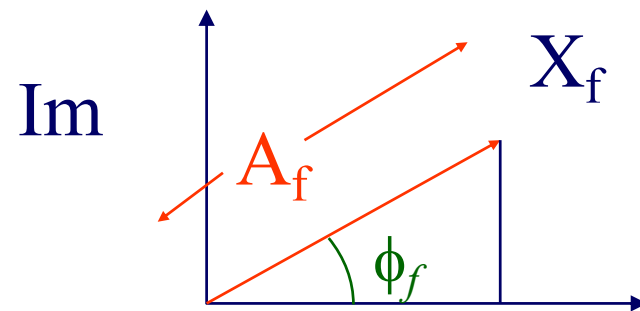
$$X_f = (X_{n-f})^*$$

(“*” : complex conjugate: $(a + bj)^* = a - bj$)

DFT: definition

Definitions

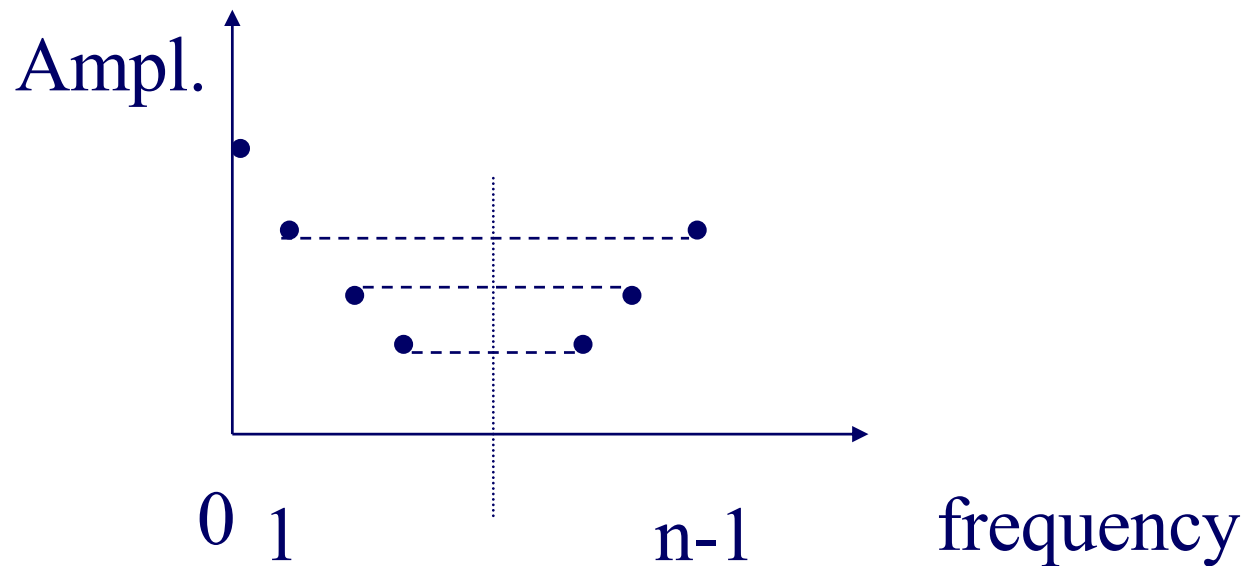
- $A_f = |X_f|$: amplitude of frequency f
- $|X_f|^2 = \text{Re}(X_f)^2 + \text{Im}(X_f)^2 = \text{energy of frequency } f$
- phase ϕ_f at frequency f



DFT: definition

Amplitude spectrum: $|X_f|$ vs f ($f=0, 1, \dots, n-1$)

SYMMETRIC (Thus, we plot the **first** half only)



DFT: definition

Phase spectrum $|\phi_f|$ vs f ($f=0, 1, \dots, n-1$):

Anti-symmetric

(Rarely used)

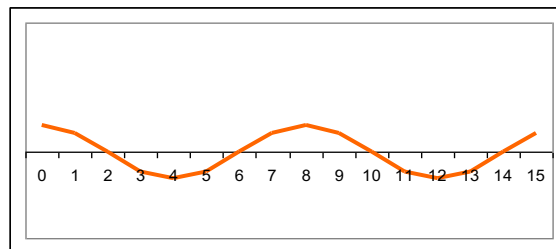
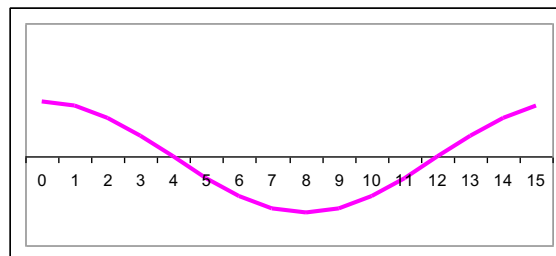
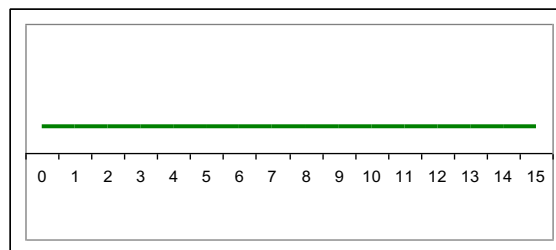
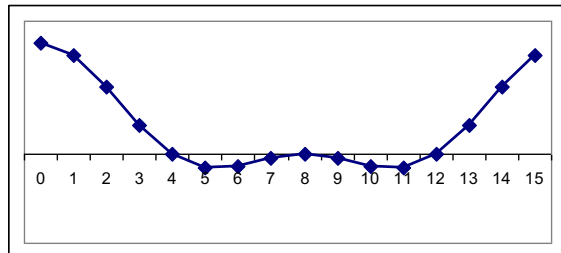


DSP - Detailed outline

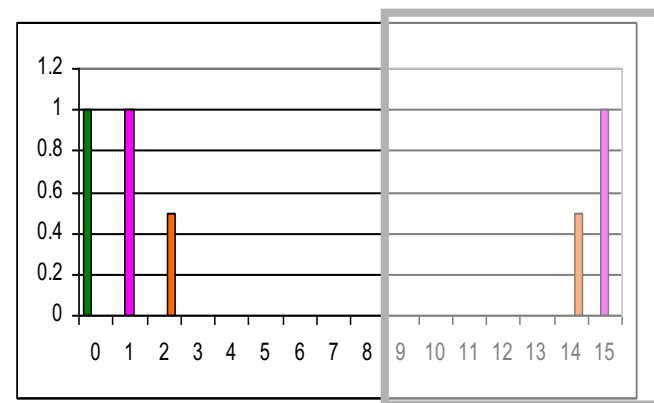
- DFT
 - what
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 - ➔ – Arithmetic examples
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DFT: examples

examples



Ampl.

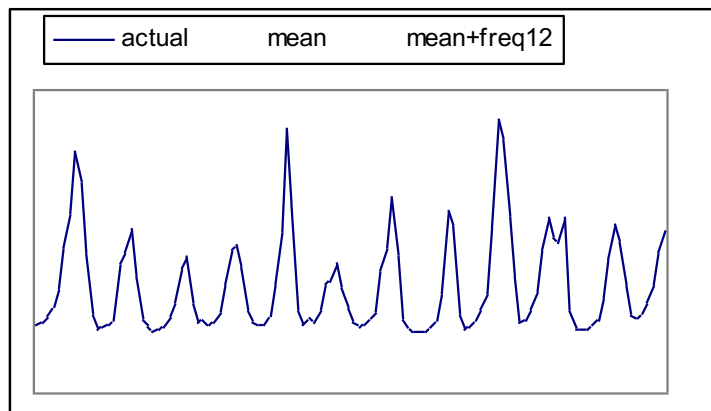


Freq.

DFT: Amplitude spectrum

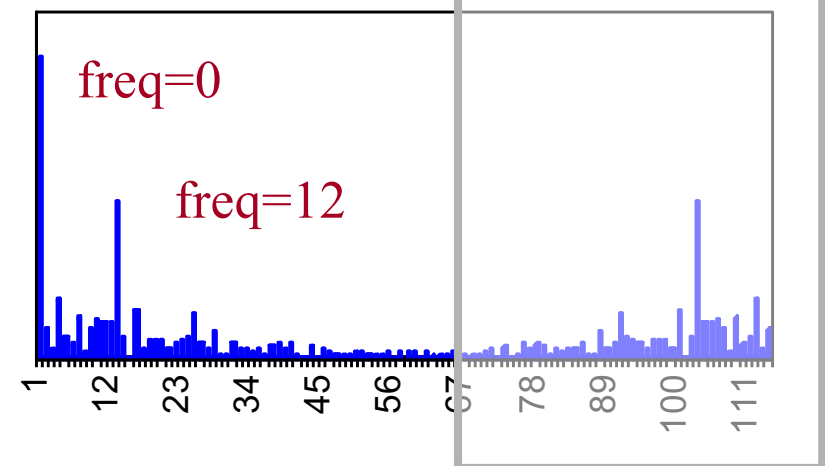
Amplitude: $A_f^2 = \text{Re}^2(X_f) + \text{Im}^2(X_f)$

count



year

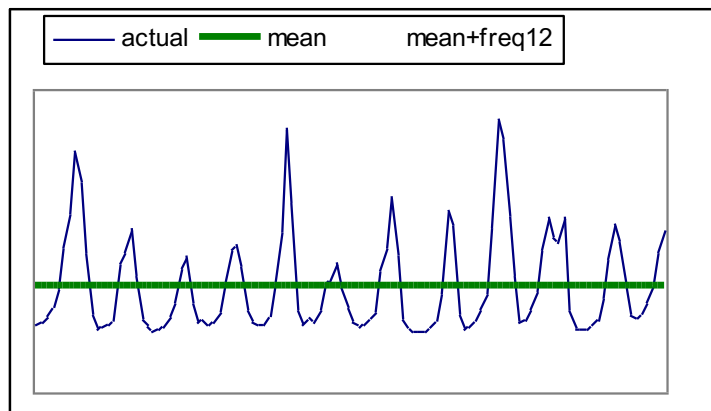
Ampl.



Freq.

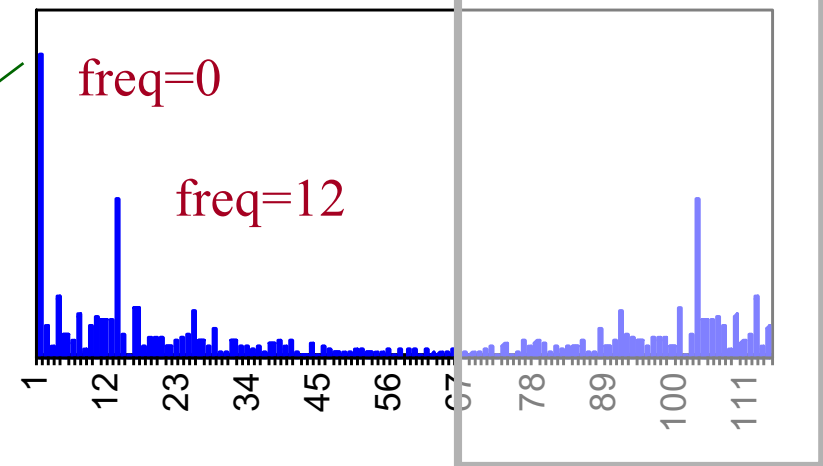
DFT: Amplitude spectrum

count



year

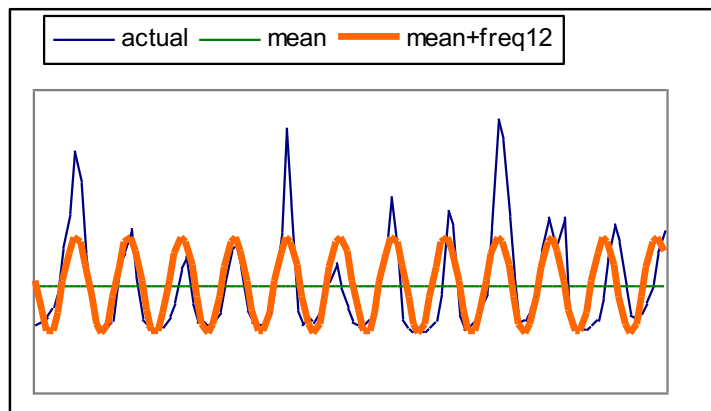
Ampl.



Freq.

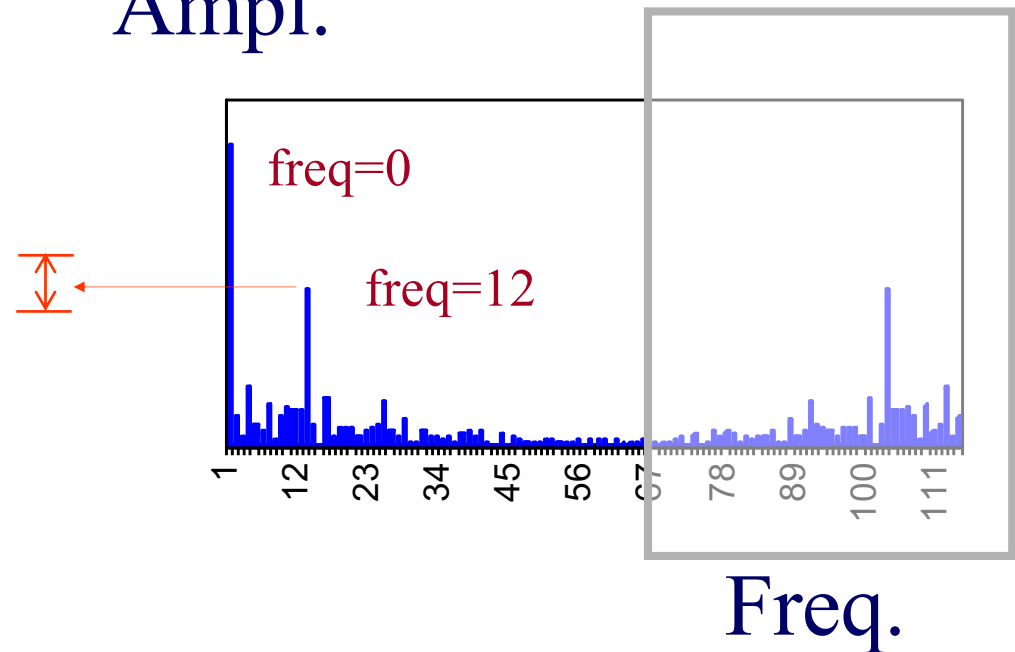
DFT: Amplitude spectrum

count



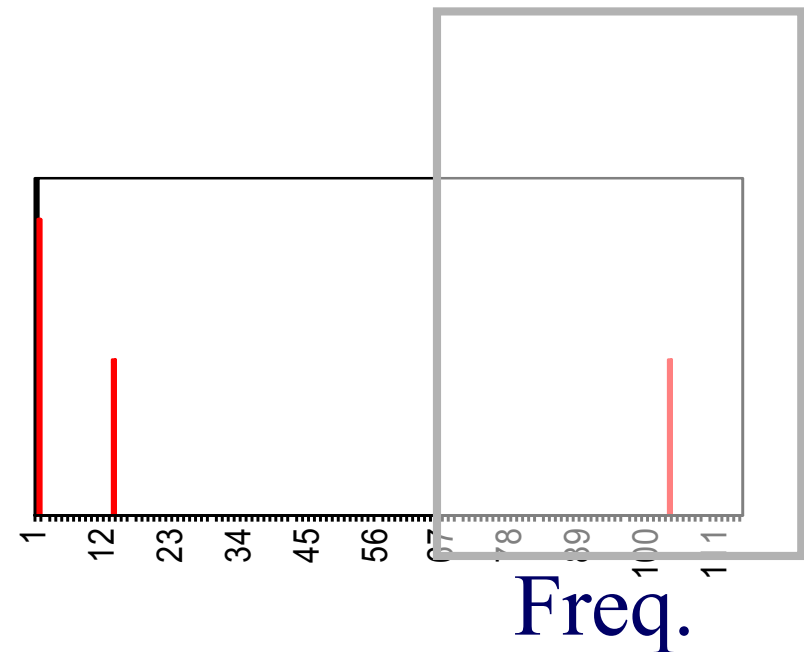
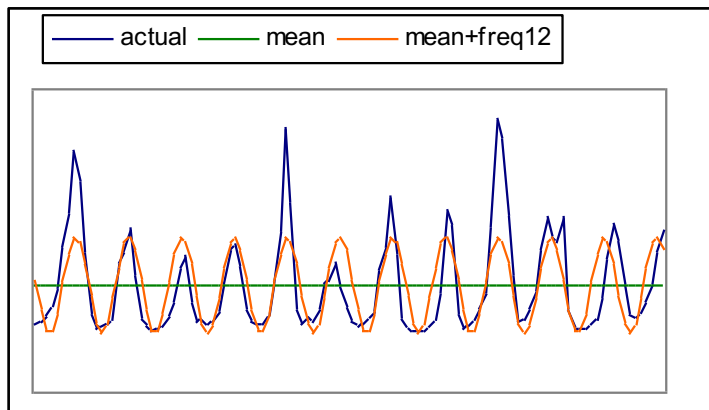
year

Ampl.



DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?

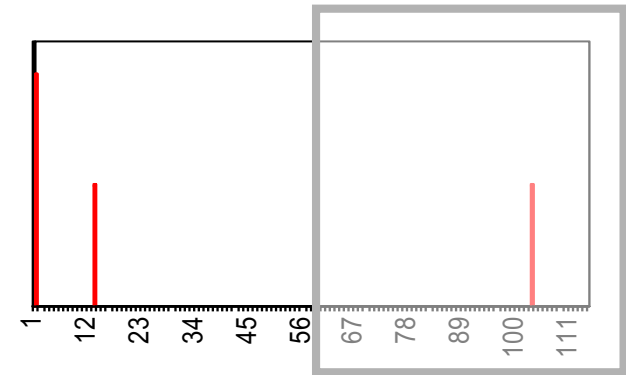


DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: compression
- A2: pattern discovery
- (A3: forecasting)

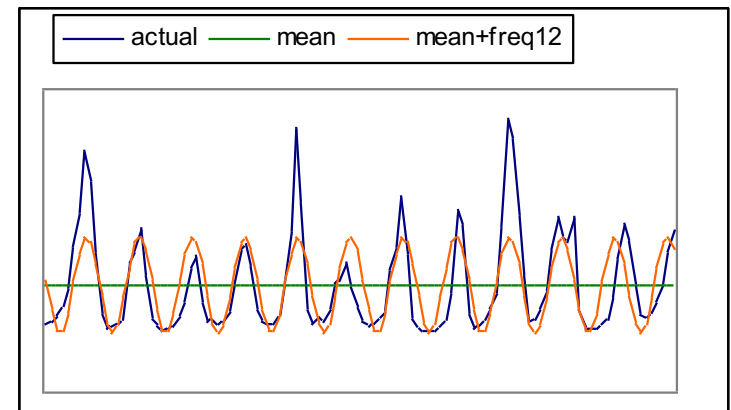
DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: **(lossy) compression**
- A2: pattern discovery



DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: (lossy) compression
- A2: **pattern discovery**





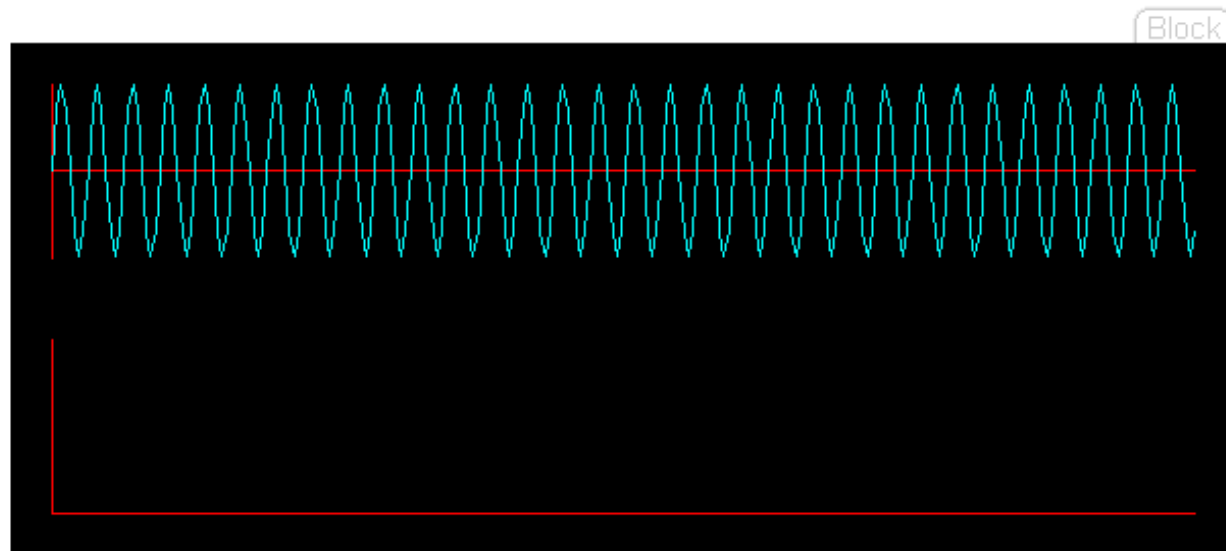
DSP - Detailed outline

- DFT
 - what
 - why
 - how
 - ➔ – Arithmetic examples
 - properties / observations
 - DCT
 - 2-d DFT
 - Fast Fourier Transform (FFT)

DFT: Amplitude spectrum

- Let's see it in action (defunct now...)
- (<http://www.dsptutor.freeuk.com/jsanalyser/FFTSpectrumAnalyser.html>)
- plain sine
- phase shift
- two sine waves
- the 'chirp' function
- <http://ion.researchsystems.com/>

Plain sine



Number of samples:

256

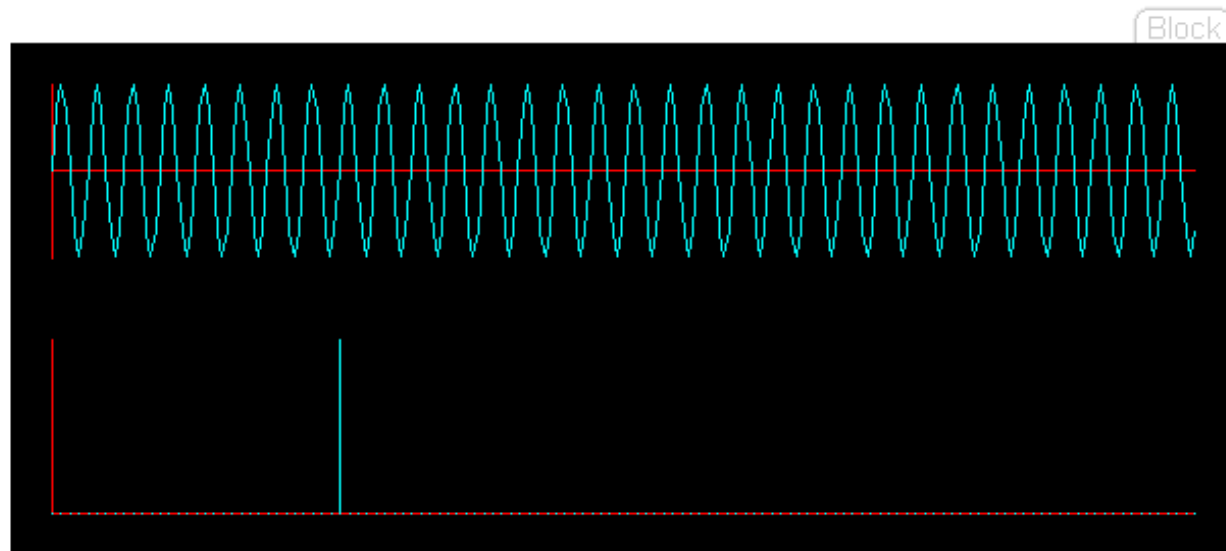
Sampling rate:

8000 samples / s

Signal waveform expression:

$\sin(2000 \cdot \pi \cdot t)$

Plain sine



Number of samples:

256

Sampling rate:

8000 samples / s

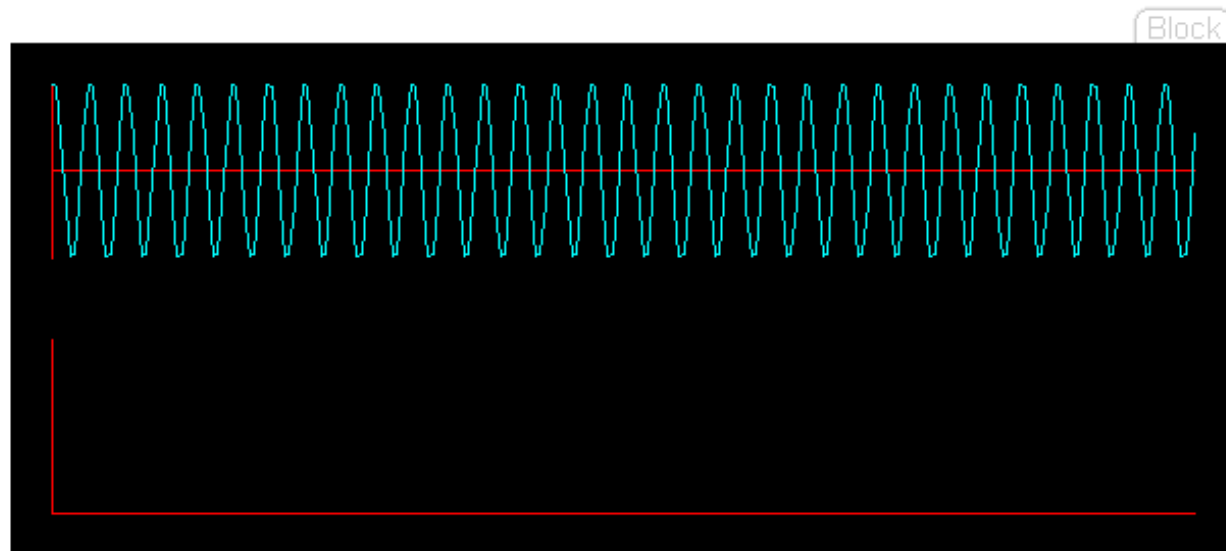
Signal waveform expression:

$\sin(2000 \cdot \pi \cdot t)$

Plot signal

Plot spectrum

Plain sine – phase shift



Number of samples:

256

Sampling rate:

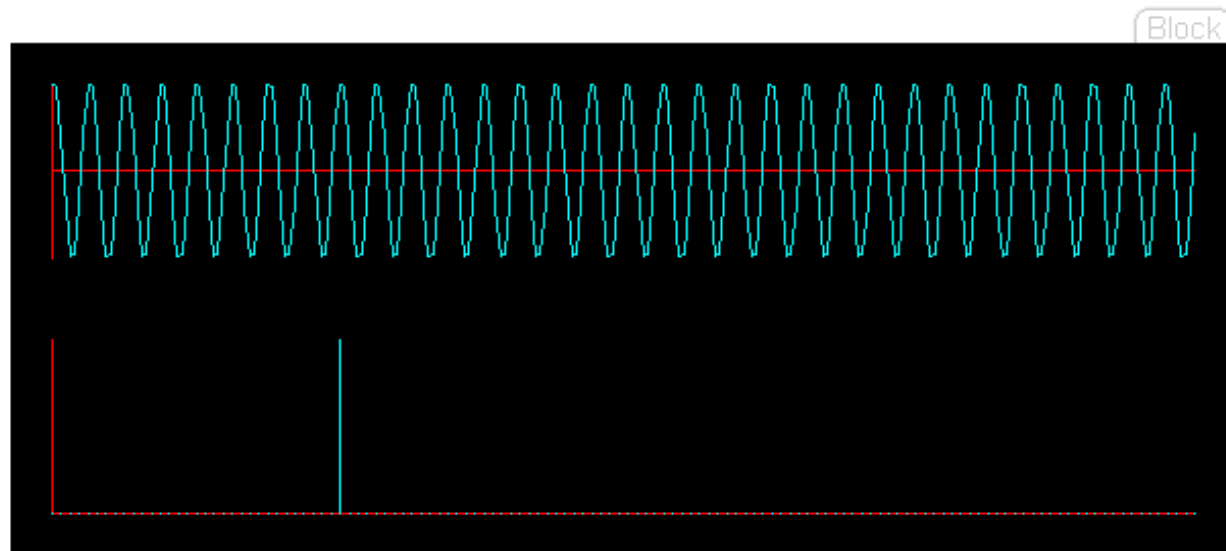
8000 samples / s

Signal waveform expression: $\sin(2000\pi t + 1.2)$

Plot signal

Plot spectrum

Plain sine – phase shift



Number of samples:

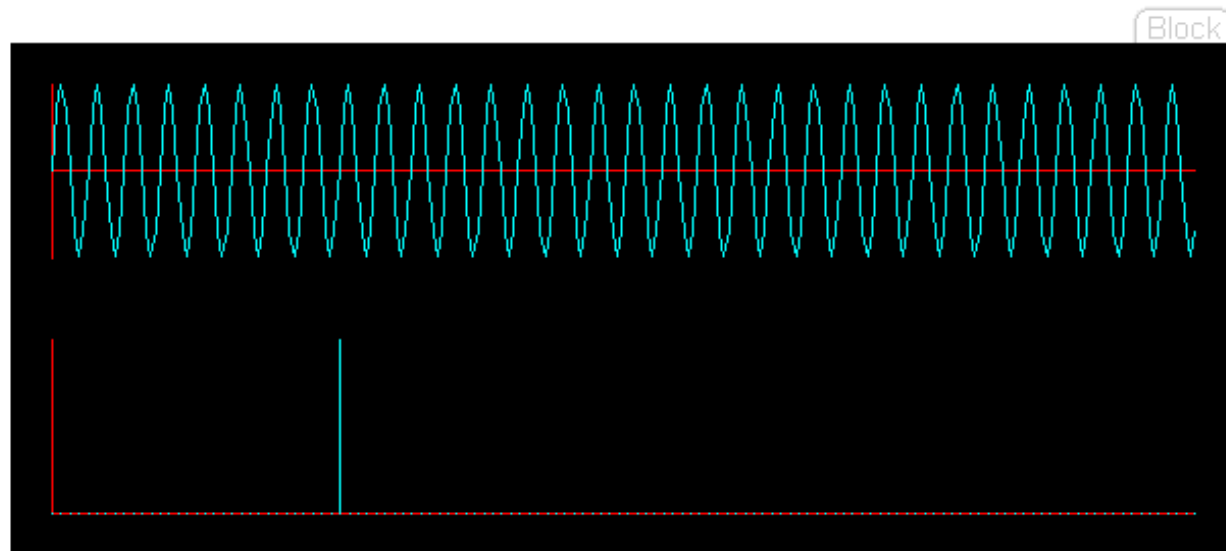
256

Sampling rate:

8000 samples / s

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Plain sine



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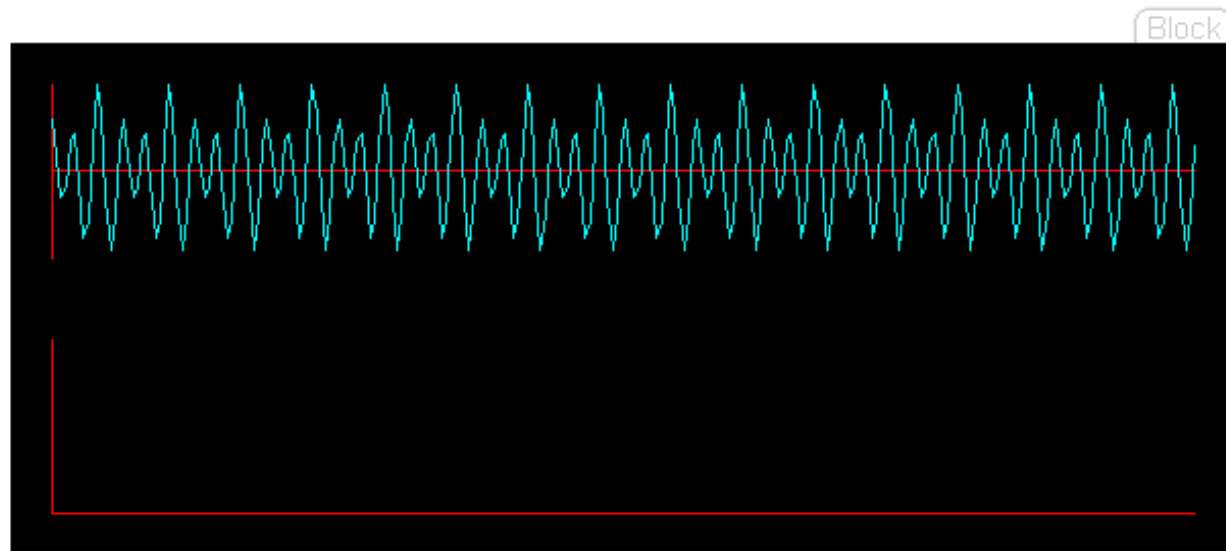
256

Sampling rate:

8000 samples / s

Signal waveform expression:

Two sines



Number of samples:

256 ▾

Sampling rate:

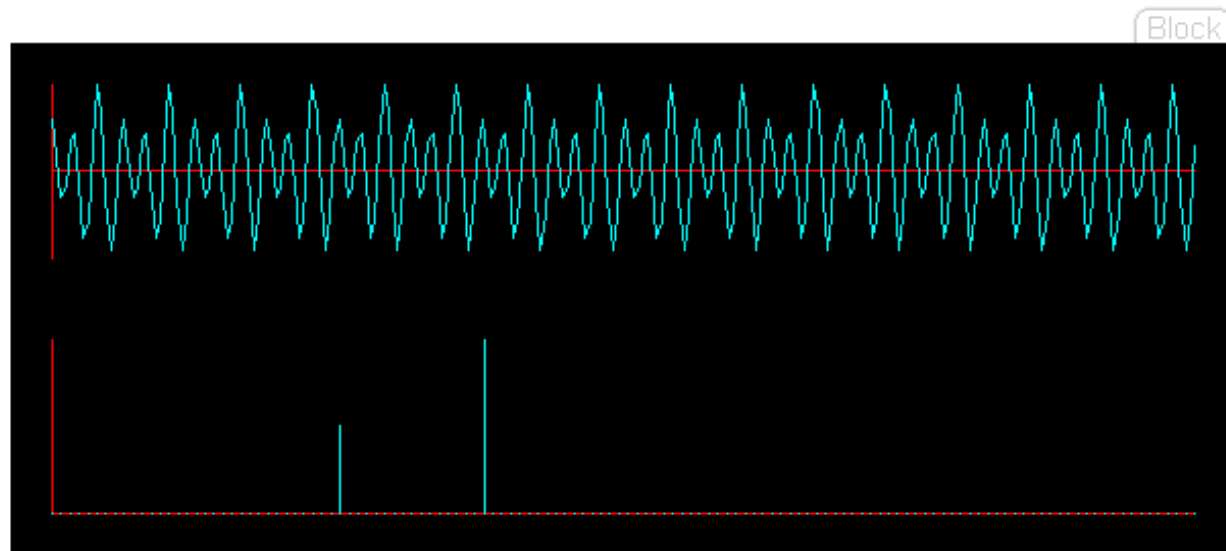
8000 samples / s

Signal waveform expression: $\sin(2000 \cdot \pi \cdot t) + 2 \cdot \cos(3000 \cdot \pi \cdot t + 0.5)$

Plot signal

Plot spectrum

Two sines



Number of samples:

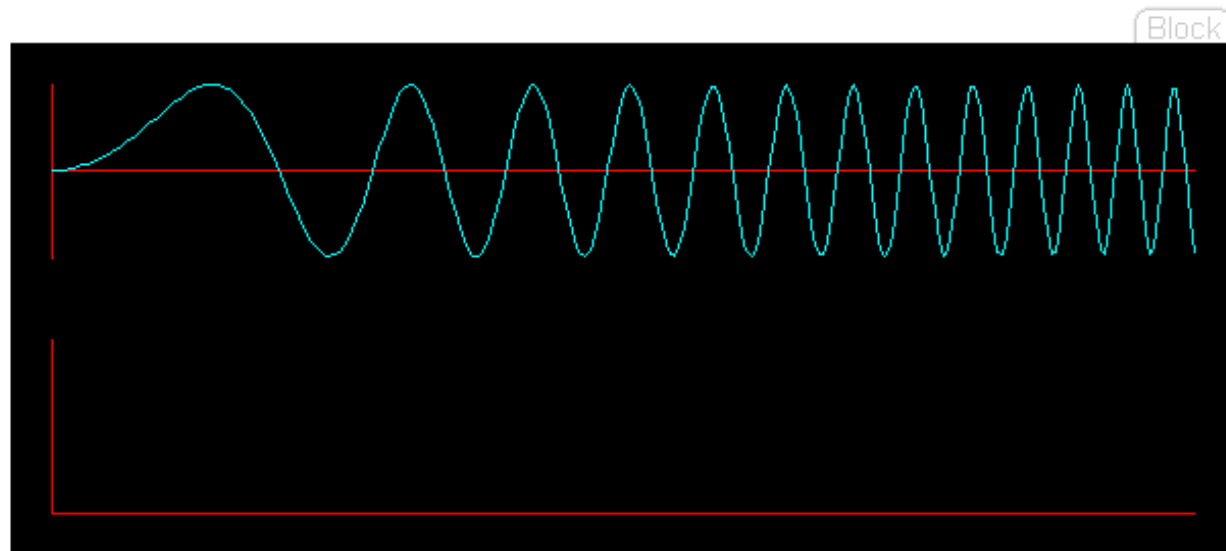
256

Sampling rate:

8000 samples / s

Signal waveform expression: $\sin(2000\pi t) + 2\cos(3000\pi t + 0.5)$

Chirp



Number of samples:

256

Sampling rate:

8000 samples / s

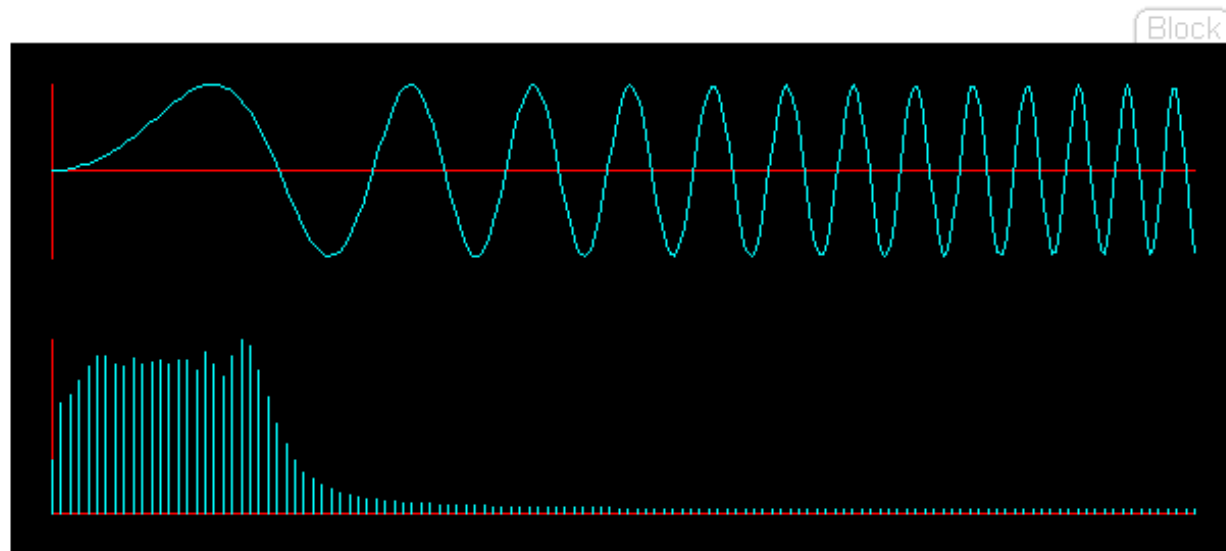
Signal waveform expression:

$\sin(25000 \cdot \pi \cdot t)$

Plot signal

Plot spectrum

Chirp



Number of samples:

256

Sampling rate:

8000 samples / s

Signal waveform expression:

$\sin(25000 \cdot \pi \cdot t^2)$

Plot signal

Plot spectrum



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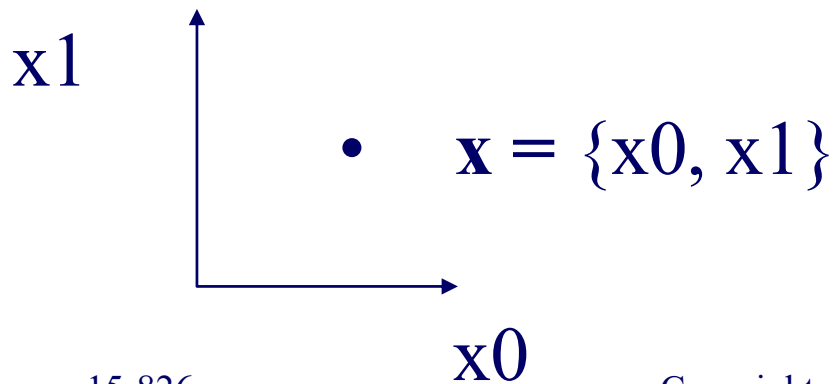
Summary of properties

- ✓ Time shift $\rightarrow$ same amplitude spectrum
- ✓ Single/dominant sinusoid: $\rightarrow$ spike in spectrum
- DFT preserves energy (Parseval's thm)
- Sawtooth: all frequencies (with decreasing ampl.)
- Spike in time: $\rightarrow$ all frequencies
- Step/Trend: $\rightarrow$ ringing ($\sim$ all frequencies)

DFT: Parseval's theorem

$$\sum (x_t^2) = \sum (|X_f|^2)$$

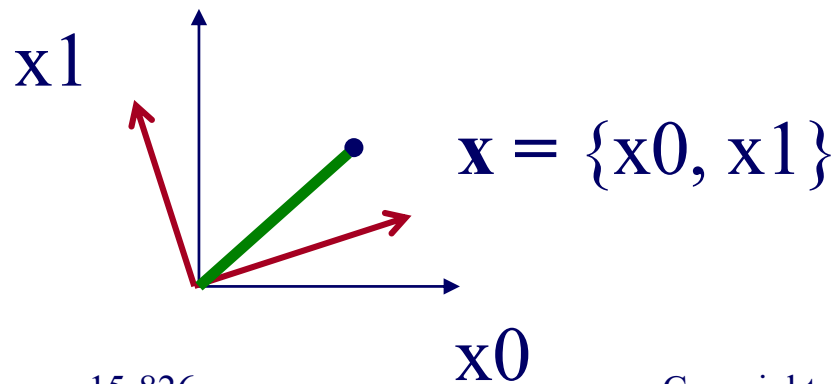
Ie., DFT preserves the 'energy'
or, alternatively: it does an axis rotation:



DFT: Parseval's theorem

$$\sum (x_t^2) = \sum (|X_f|^2)$$

Ie., DFT preserves the 'energy'
or, alternatively: it does an axis rotation:



DFT: Parseval's theorem

$$\sum (x_t)^2 = \sum (|X_f|^2)$$

... equivalently,

matrix $\mathbf{F} (= \left[\frac{1}{\sqrt{n}} e^{-j2\pi f t} \right])$

is row-orthonormal

Row: f

Column: t

$$\begin{matrix} X_0 \\ X_1 \\ \vdots \\ X_f \\ \vdots \end{matrix} \begin{matrix} \bigg| \\ \bigg| \\ \bigg| \\ \bigg| \\ \bigg| \end{matrix} = \begin{matrix} & \mathbf{F} \\ \begin{matrix} \boxed{e^{-j2\pi f t}} \end{matrix} & \begin{matrix} \bigg| \\ \bigg| \\ \bigg| \\ \bigg| \\ \bigg| \end{matrix} \end{matrix} \begin{matrix} x_0 \\ x_1 \\ \vdots \\ x_t \\ \vdots \end{matrix}$$



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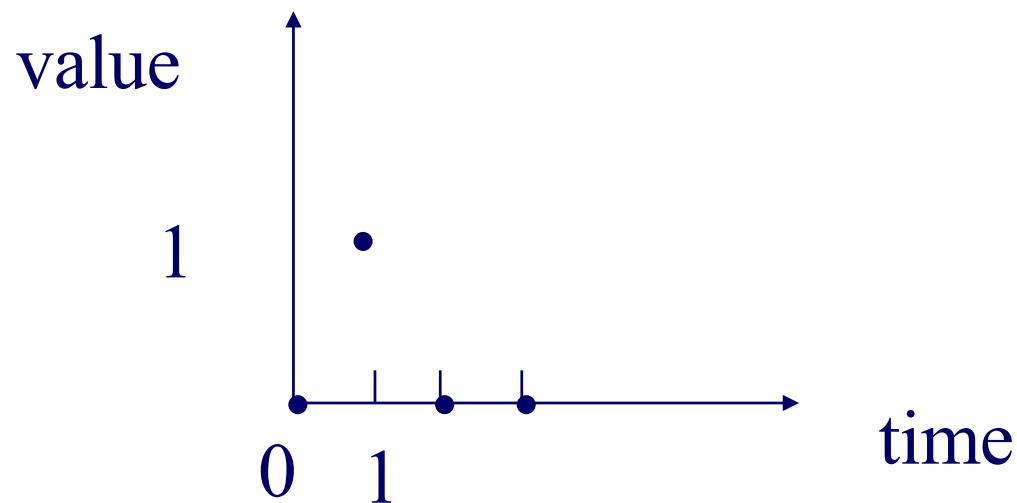


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Arithmetic examples

- Impulse function: $\mathbf{x} = \{0, 1, 0, 0\}$ ($n = 4$)
- $X_0 = ?$



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- A: $X_0 = 1/\text{sqrt}(4) * 1 * \exp(-j 2 \pi 0 / n) = 1/2$
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- Q: does the 'symmetry' property hold?

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 - Q: does the 'symmetry' property hold?
 - A: Yes (of course)
- 

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- $X_3 = +1/2 j$
- Q: check Parseval's theorem

Arithmetic examples

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- $X_3 = +1/2 j$
- Q: (Amplitude) spectrum?
- A: FLAT!

Arithmetic examples

- Q: What does this mean?

Arithmetic examples

- Q: What does this mean?
- A: All frequencies are equally important ->
 - we need n numbers in the frequency domain to represent just one non-zero number in the time domain!
 - “*frequency leak*”

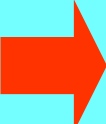


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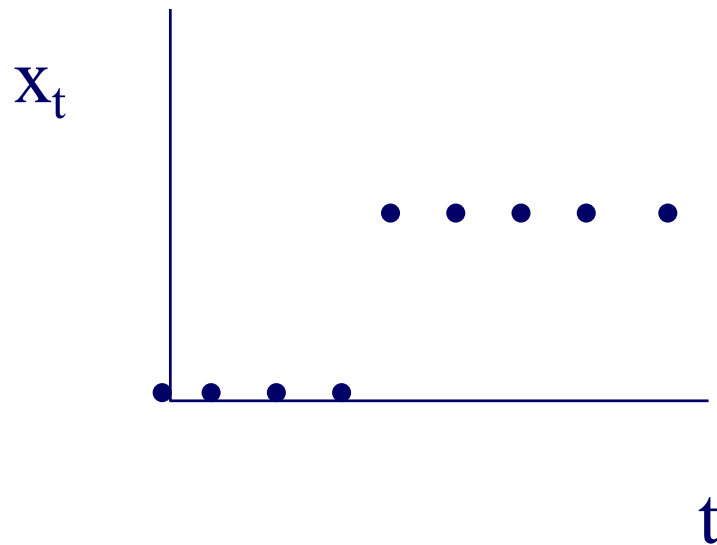


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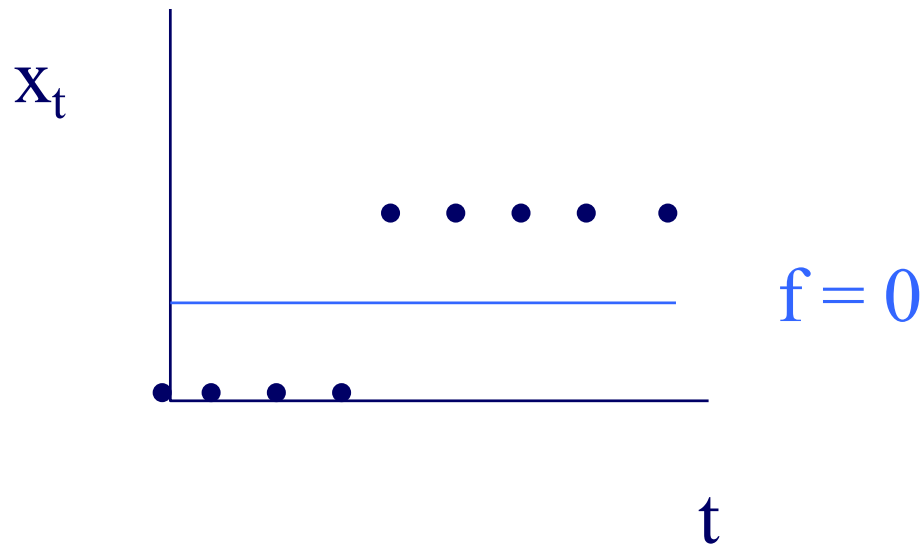
Observations

- DFT of ‘step’ function:
 $x = \{ 0, 0, \dots, 0, 1, 1, \dots 1 \}$



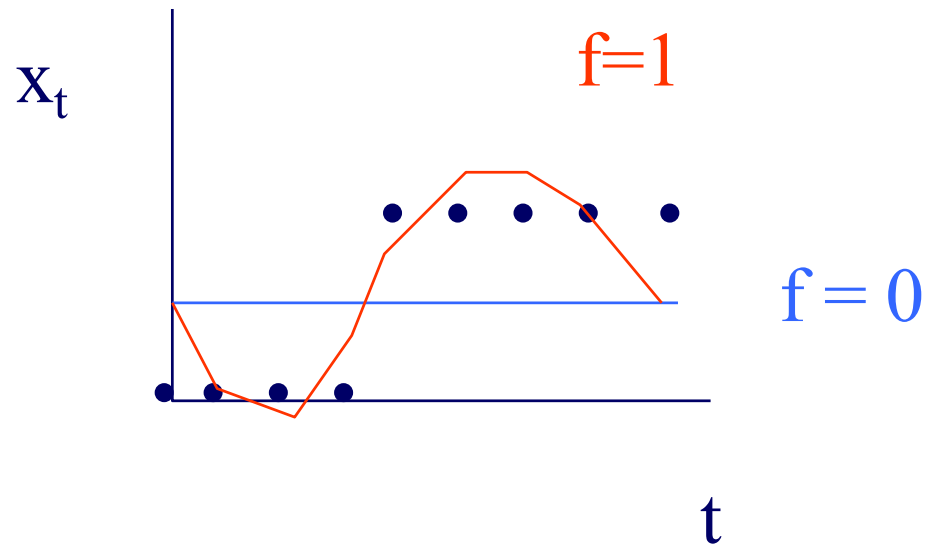
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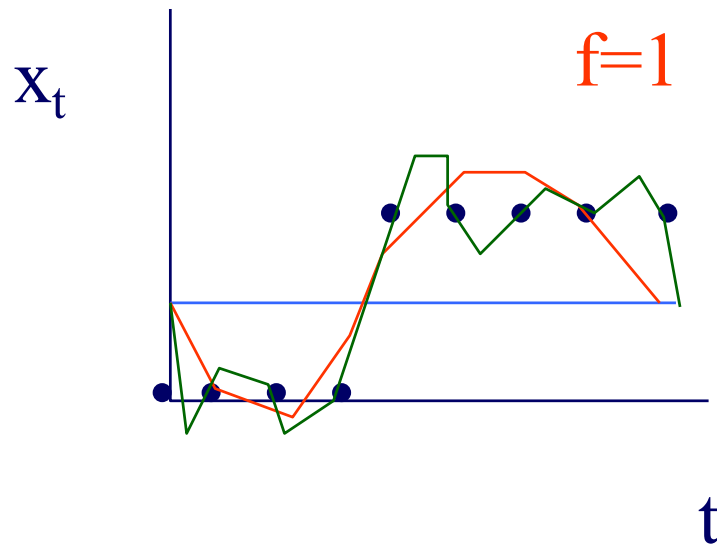
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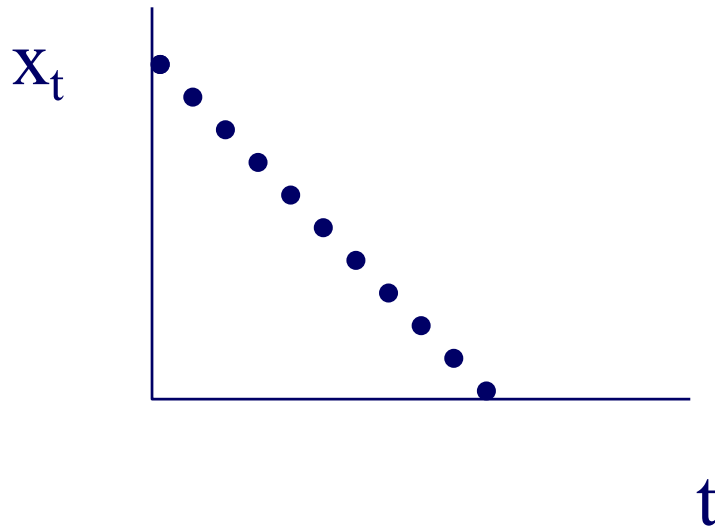
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- the more frequencies,
the better the approx.
- ‘ringing’ becomes worse
- reason: discontinuities; trends

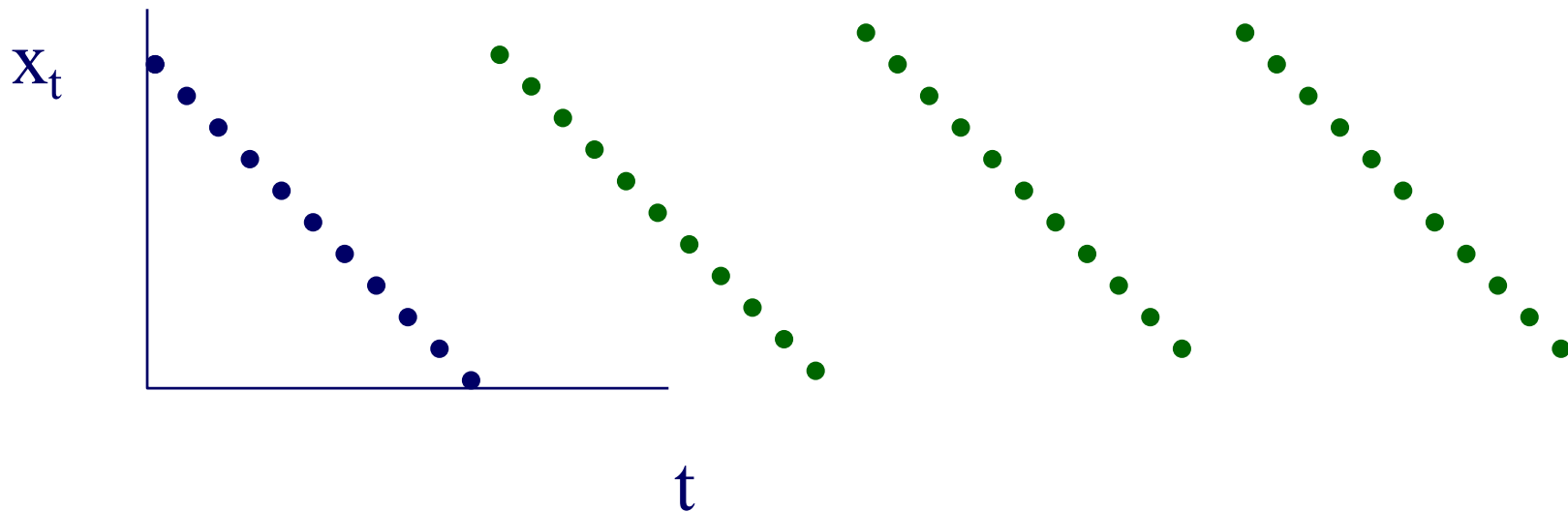
Observations

- Ringing for trends: because DFT ‘sub-consciously’ replicates the signal



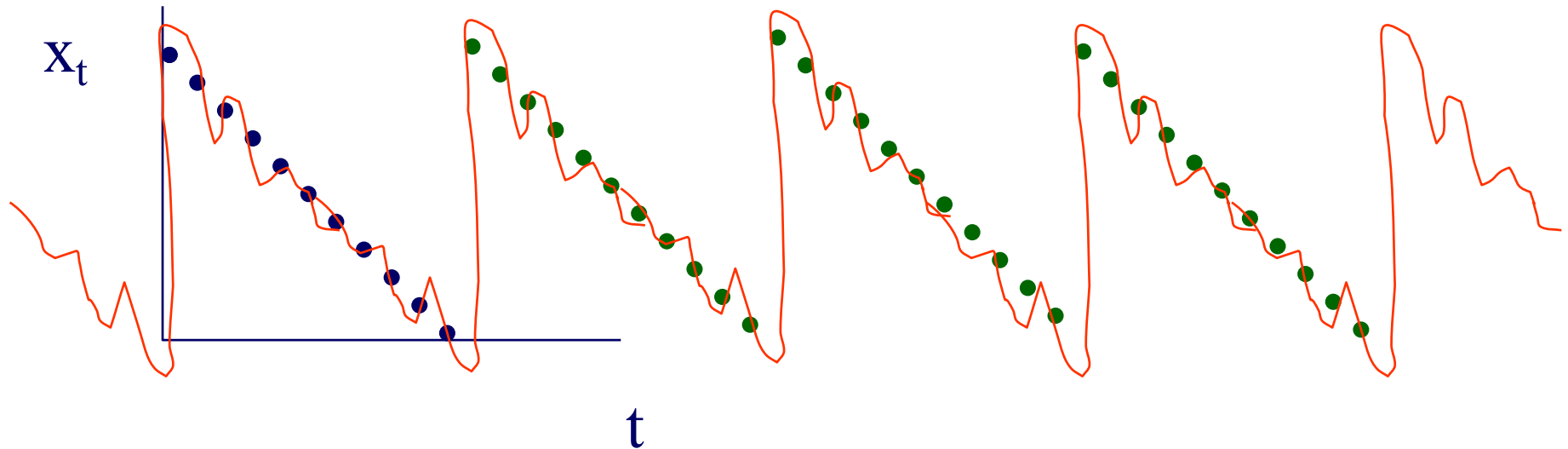
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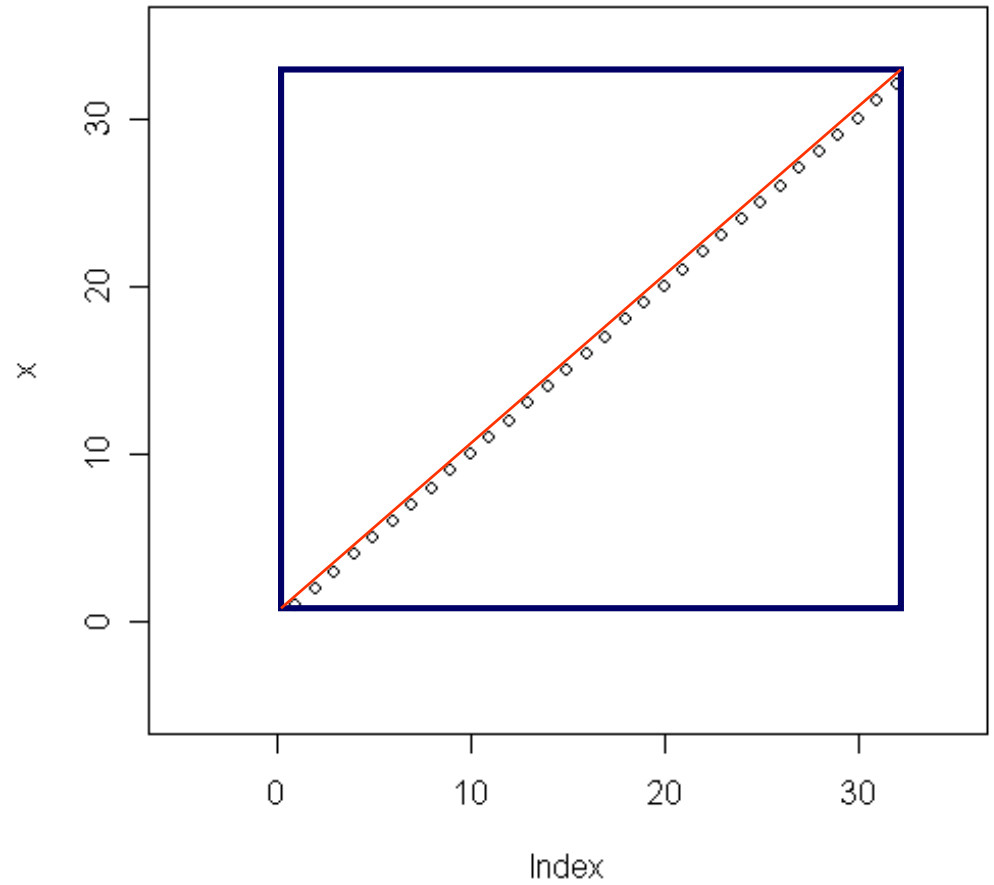


Observations

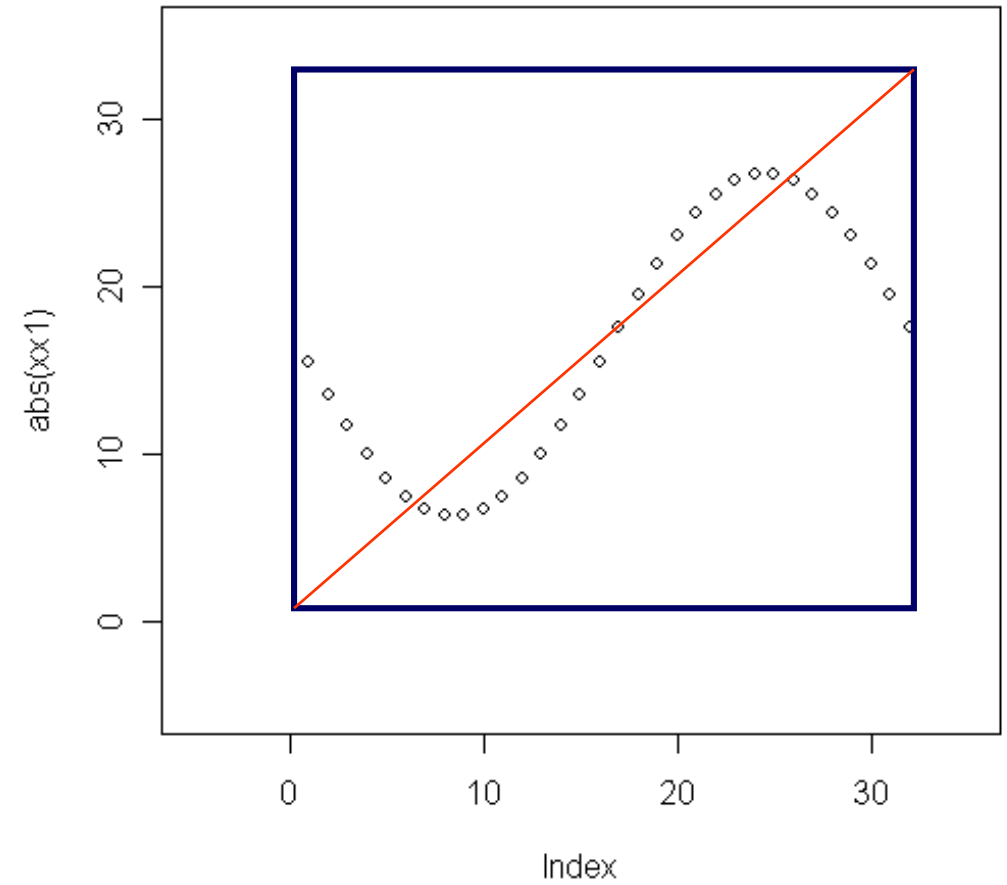
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original

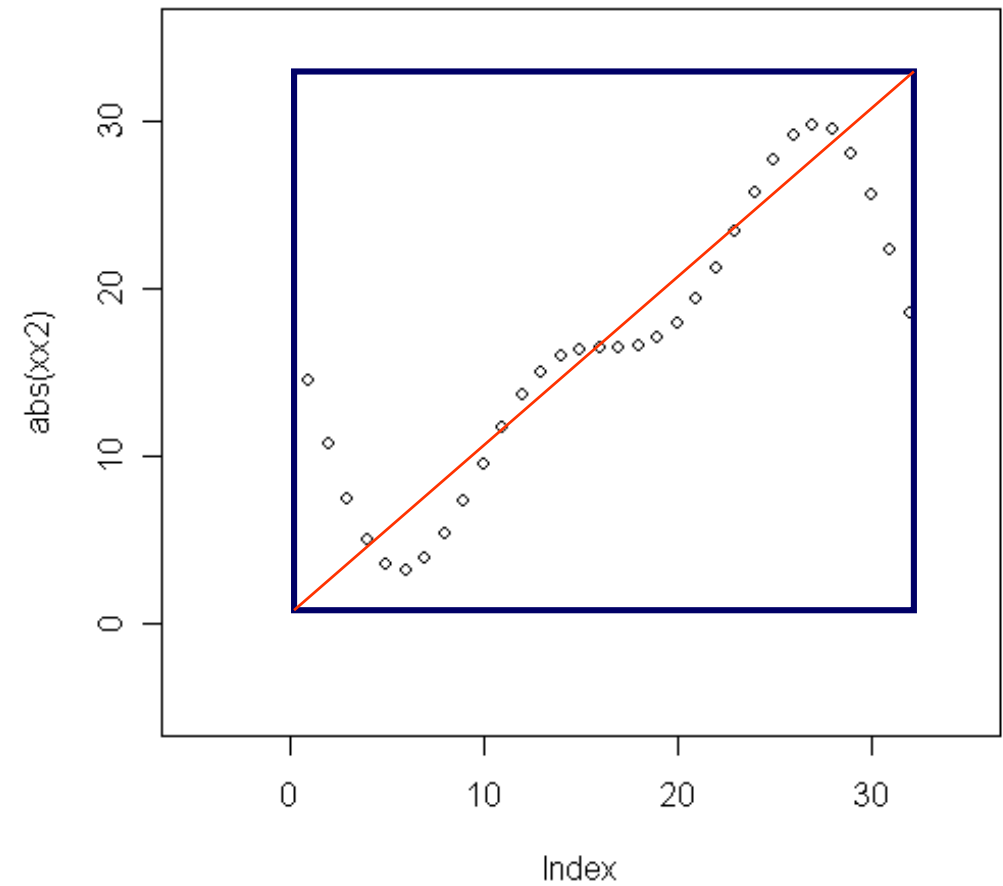


DC and 1st



DC and 1st

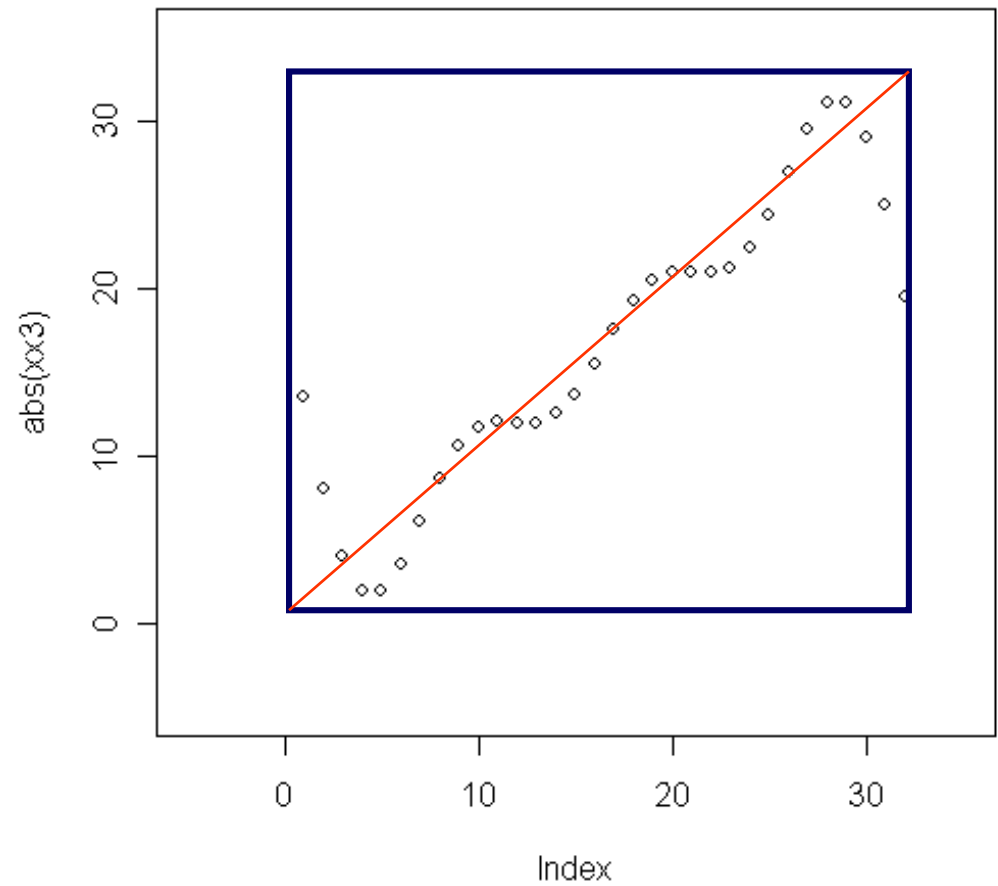
And 2nd



DC and 1st

And 2nd

And 3rd

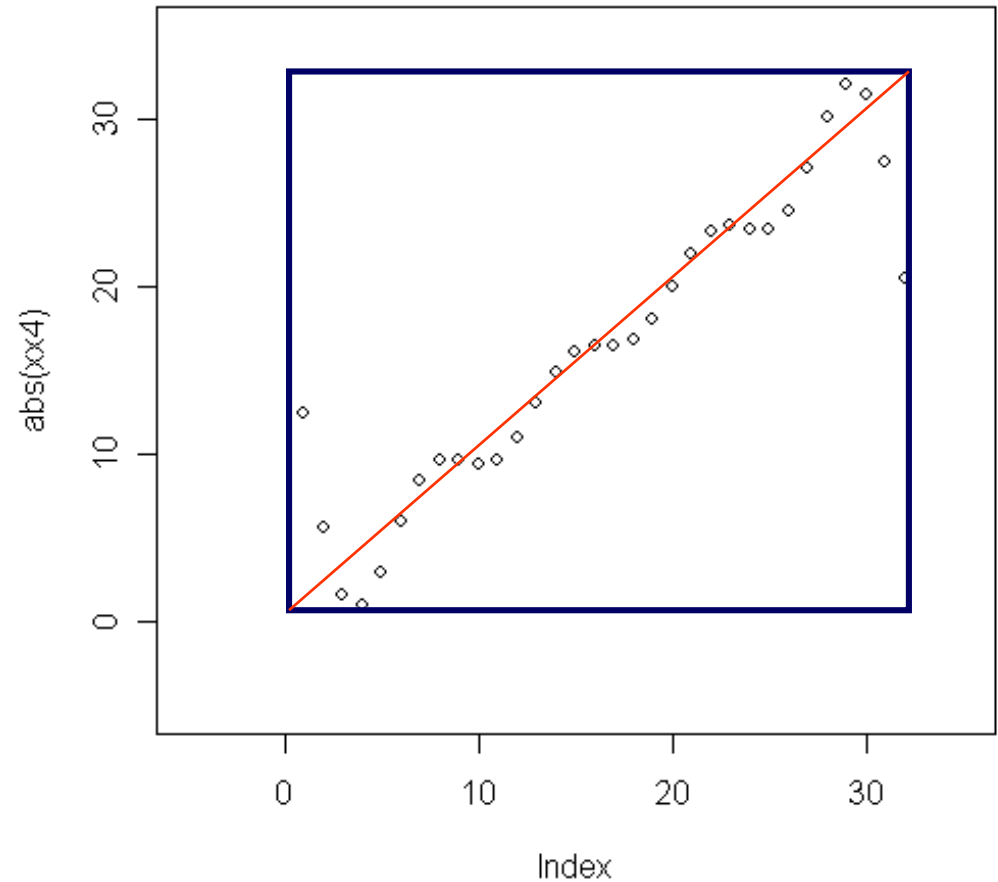


DC and 1st

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- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin(2 \pi / 4 t)$$

($t = 0, \dots, 3$)

- Q: $X_0 = ?$
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- Q: $X_0 = 0$
 - Q: $X_1 = -3j$
 - Q: $X_2 = 0$
 - Q: $X_3 = 3j$
- check 'symmetry'
 - check Parseval

Observations

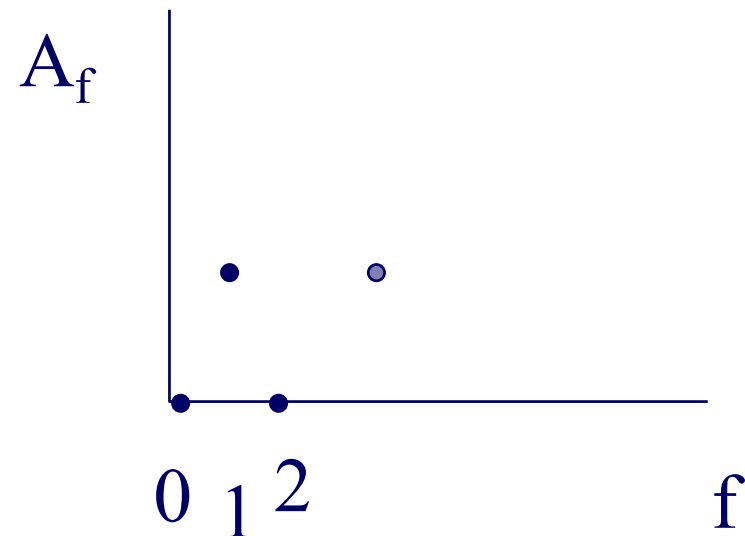
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- Does this make sense?





A: Yes, it makes sense:

- ✓ Time shift $\rightarrow$ same amplitude spectrum
- ➔ ✓ Single/dominant sinusoid: $\rightarrow$ spike in spectrum
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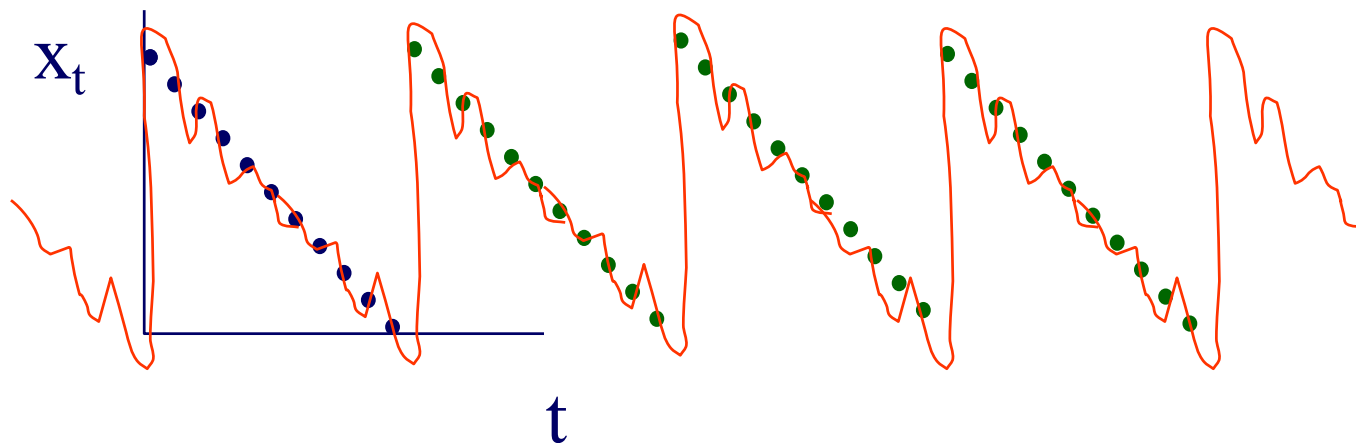
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DCT

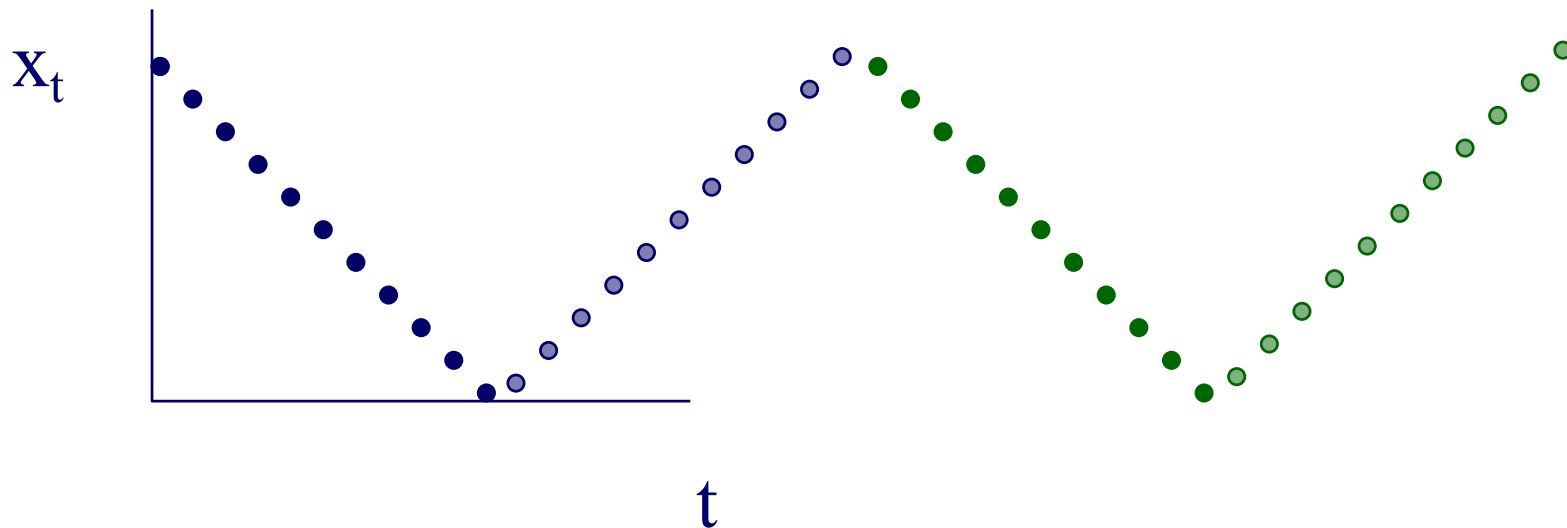
Discrete Cosine Transform

- motivation#1: DFT gives complex numbers
- motivation#2: how to avoid the ‘frequency leak’ of DFT on trends?



DCT

- brilliant solution to both problems: mirror the sequence, do DFT, and drop the redundant entries!



DCT

- (see Numerical Recipes for exact formulas)

DCT - properties

- it gives real numbers as the result
- it has no problems with trends
- it is very good when x_t and $x_{(t+1)}$ are correlated

(thus, is used in JPEG, for image compression)



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2-d DFT

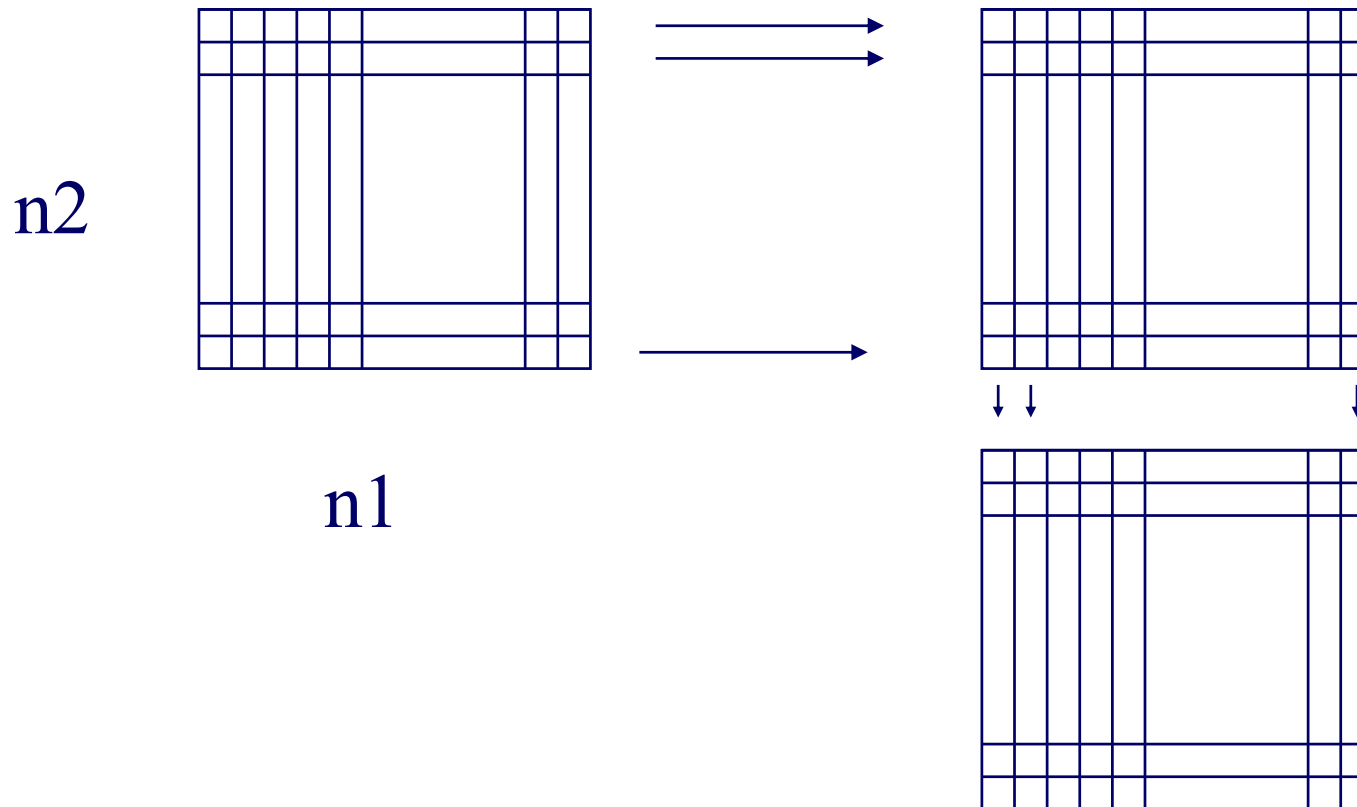
- Definition:

$$X_{f_1, f_2} = \frac{1}{\sqrt{n_1}} \frac{1}{\sqrt{n_2}} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} x_{i_1, i_2} \exp(-2\pi j i_1 f_1 / n_1) \exp(-2\pi j i_2 f_2 / n_2)$$

2-d DFT

- Intuition:

do 1-d DFT on each row



and then
1-d DFT
on each
column

2-d DFT

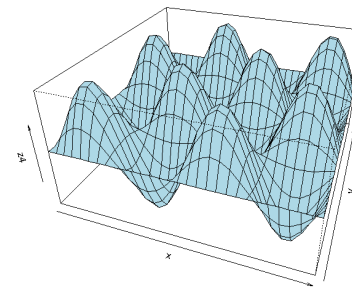
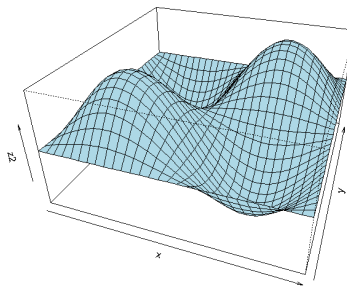
- Quiz: how do the basis functions look like?
- for $f_1 = f_2 = 0$
- for $f_1 = 1, f_2 = 0$
- for $f_1 = 1, f_2 = 1$

2-d DFT

- Quiz: how do the basis functions look like?
- for $f_1 = f_2 = 0$ flat
- for $f_1 = 1, f_2 = 0$ wave on x; flat on y
- for $f_1 = 1, f_2 = 1$ ~ egg-carton

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FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi tf / n)$$

FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi tf / n)$$

- A: Naively, $O(n^2)$

FFT

- However, if n is a power of 2 (or a number with many divisors), we can make it

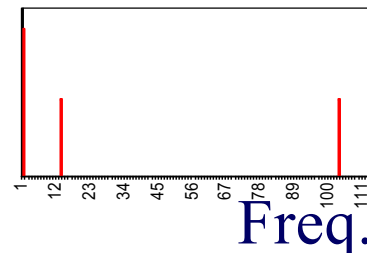
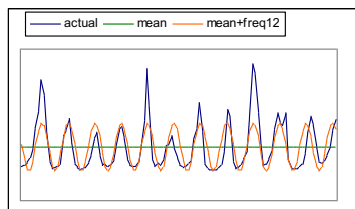
$$O(n \log n)$$

Main idea: if we know the DFT of the odd time-ticks, and of the even time-ticks, we can quickly compute the whole DFT

Details: in Num. Recipes

DFT - Conclusions

- It spots periodicities (with the ‘**amplitude spectrum**’)
- can be quickly computed ($O(n \log n)$), thanks to the FFT algorithm.
- **standard** tool in signal processing (speech, image etc signals)





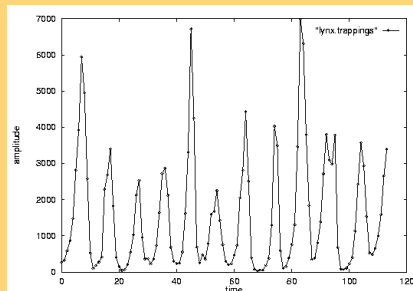
Solutions:

Goal: given a signal (eg., sales over time and/or space)

Q: Find patterns and/or compress

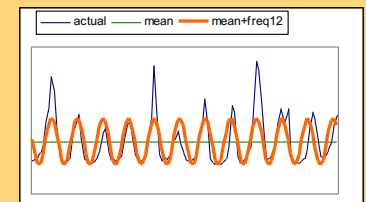


count

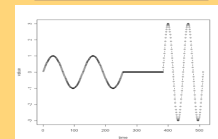
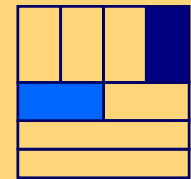


year

✓ A1: Fourier (DFT)



A2: Wavelets (DWT)





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